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New Perspectives on Mathematical Proof

In mathematics, one seeks a framework for the mind which compels logical thought and demands adherence (whether direct or indirect) to some fundamental axiom set. In many instances, it is the general goal of mathematics to crystallize an otherwise complex process[1] into a few basic principles, then derive from those principles, via a well-defined method, a series of truths which, if the particular theory is successful[2], will aid in the understanding of that process. Turning to what is often called pure mathematics, one finds that a similar procedure of "truth-making" is followed even when it is one's desire to know entirely platonic realms—processes entirely independent of any physical reality. Herein lies the great power and utility of mathematics. Whether or not the connection between abstraction and reality which is inevitably established is permissible is beyond the scope of this paper; however, it should be noted that such a connection will be assumed to be valid and real.[3]

Having ascribed such a powerful interpretive role to mathematics (and indeed, one finds that the field is absolutely indispensable in the modern world), it is only natural to shift attention to the concept of proof. Although a given mathematical interpretation may "miss the mark" with regard to explaining some natural phenomenon (i.e. one interpretation may be better than another), it must be the case that the truths produced from any formal system follow logically from the premises. Were mathematics to allow the derivation of falsehoods, it would be a useless field. Furthermore, it is often the case that the answers to interesting[4] mathematical questions are not obviously related to the axioms or known truths of a system. It is critical, therefore, that there exist methods by which the truth value of a given proposition may be conclusively determined. In this paper, these methods will be referred to as proofs, where proof is defined as follows:

> *To show conclusively that a given proposition is a truth corresponding to some formal system is to prove it; to show that it is a falsehood is to disprove it.*

Proof, then, makes possible advancement in mathematics, for it is the vehicle by which new truths are verified and, in some respect, added to the known body of true propositions. What is unclear, however, is the manner in which proofs are carried-out, how to handle the case of a proposition being undecidable, and why some proofs are deemed more valuable than others.

Given an interesting mathematical question, one might prove or disprove it using a variety of techniques. Although a comprehensive overview of the myriad ways in which

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[1] Here the term *complex process* is meant to indicate some behavior, phenomenon, etc. about which no conclusive statements could otherwise be made.

[2] A theory may be deemed successful based on how closely it agrees with empirical evidence.

[3] See, for example, Pincock, Christopher. *A New Perspective on the Problem of Applying Mathematics* (2003) for more on this issue.

[4] By *interesting* questions, the author implies those questions whose answers are either not obvious or lead to greater truths, perhaps in other branches of mathematics.

a proof may be approached will not be given here, several predominant methods should be mentioned. Making a distinction between existential and universal statements, it is possible to immediately consider two types of proof. A statement of the form $\exists x \in D \mid Q(x)$, for example, may be proved by providing some x in the specified domain D which makes $Q(x)$ (a predicate) true. Additionally, one might provide an algorithm for finding such an x. Proofs of this form are deemed *constructive*, as opposed to *nonconstructive* proofs, which simply employ axioms or known truths guaranteeing the truth of the statement at hand. Universal statements of the form $\forall x \in D, P(x) \rightarrow Q(x)$ may also be proved via two general techniques. If the domain D in question is finite, a proof by *exhaustion* might be considered. Such a proof entails showing explicitly (by enumeration) that the condition "if $P(x)$ then $Q(x)$" holds for every x within D. If D is infinite, however, a more powerful technique must be used. What is sought in such cases is the *method of generalizing from the generic particular*, which involves the manipulation of a generic x—to be taken as some particular but arbitrary member of D—until the associated condition is satisfied (Epp, 115-7). By generalizing all possible members of D to the generic element x, whose properties must coincide with the properties of any element of D, one effectively operates on a potentially infinite class of objects simultaneously, and may therefore produce a conclusive statement about all such objects.

Such methods, along with the many specific proof techniques derived from them, leave one well-equipped to determine the truth or falsity of any proposition; nonetheless, a clear problem presents itself. How might one, for example, deal with the case of an undecidable proposition?[5] In his famous 1930 address, Hilbert made clear his belief that there exist no unsolvable mathematical problems, declaring, in a spirit of great optimism, *wir müssen wissen, wir werden wissen*[6]. Whether such an attitude is justified continues to be a matter of debate. Gödel proved via his incompleteness theorem that undecidable propositions do in fact present themselves in elementary arithmetic, but it is not clear whether they arise in other branches of mathematics. As Franzén points out, such a determination is difficult to make, since the axioms of a theory may be formally extended (e.g. by reflection principles). This implies that even though certain axioms may guarantee the truth of some proposition, "we may have no reason or inclination to accept the axiom"(14).

These problems aside, attempting to prove that a proposition is undecidable raises an interesting philosophical conundrum. Because such a proposition can, by definition, be proved neither true nor false, it seems that some fundamentally new method of proof would be necessary to show that a problem is unsolvable. Even the notion of proving undecidability runs contrary to intuition, since this may be interpreted as showing, for some proposition P, $\sim(P \vee \sim P)$—a logical contradiction.

The final consideration to be made in this paper is that of beauty in proof, or why some proofs are more desirable than others. At first glance, it is difficult to perceive any reason why one proof might be "better" than another. Holistically, the proofs of any statement are all to the same end (and therefore equivalent), and so further analysis is required. In *A Mathematician's Apology*, Hardy cites "a certain *generality* and a certain

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[5] An undecidable proposition is one whose truth value cannot be ascertained by any means, i.e. a problem which is absolutely unsolvable.

[6] "We must know, we will know"

depth"(103) as essential elements of a superior proof, although he gives fairly vague descriptions of both terms. The importance of a proof being general enough to be applicable in many branches of mathematics yet particular enough to give a powerful result is stressed. With regard to depth, Hardy writes that a beautiful proof must be deep in the sense that it offers insight into difficult and fundamental aspects of an issue: "It seems that mathematical ideas are arranged somehow in strata…The lower the stratum, the deeper (and in general the more difficult) the idea"(110).

More recently, Avigad has sought to provide stricter definitions of what constitutes superiority among proofs. He considers proofs of several well-known truths, showing, for example, how a theorem of Diophantus may be proved via simple, commutative operations of arithmetic, in terms of Gaussian integers, and through a combination of the two. Avigad also discusses a difference between strictly formal, deductive proofs and those which are written in a more informal language (although they still retain their power). He notes that the proofs shown in the majority of texts are of the informal variety, indicating perhaps that even the manner in which a proof is presented plays a role in its quality, since "'real' proofs often contain more elaborate instructions as to how one can 'see' that an assertion follows from its predecessors"(24).

Upon being given the task of determining the truth value of some proposition, one may approach the problem in a variety of ways, as has already been mentioned. Within the general framework of qualifying existential and universal statements, a mathematician might employ techniques such as deduction, induction, *reductio ad absurdum*, proving the contrapositive, etc. Strict adherence to any of these techniques, however, could very well counteract a true understanding of what is involved in proof. It was stated earlier that proof entails showing conclusively that a given proposition corresponds to some formal system. Attempting now to gain a more comprehensive, universal idea of what this means, it is worth taking the concept of a proposition to an abstract level. Let any proposition be defined as some permutation in an infinite space[7], where this space, in its ground state, represents a sort of vacuous template upon which the human intellect may operate. The adoption of axioms and the subsequent derivation of truths then represent permuting this space so as to invoke a certain structure in it. Interpreting propositions as structures within this space, one can easily envision a greater picture of what proof really is. The Riemann Hypothesis, to take an example, states that all nontrivial zeros of the Riemann zeta function lie on the real line $s = ½$. To prove such a claim is isomorphic to resolving the hypothesis into an implication $P \rightarrow Q$, where P is the set of nontrivial zeros and Q the truth condition that they must lie on the line $s = ½$, and *deconstructing* the structural representations of P and Q into an equality within the aforementioned space. Turning to a more analytical approach, one might consider (in theory only) some highly fundamental, universal language capable of expressing any proposition. As with the previous schema, proof, in this case, is isomorphic to constructing the proper relation between terms and then performing basic operations on them to arrive at an equality. If an equality is reached, the terms are indeed equivalent; if a contradiction is reached, they are not.

The case of undecidable propositions, however, has yet to be dealt with. Although it is difficult to imagine the sort of proof required to show that a given problem is

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[7] Such a representation is meant to be completely abstract and conceptual, and is not to be confused with traditional notions of space.

unsolvable, it is reasonable to assume that a fundamentally new step is needed. Because an undecidable proposition does not really lie "inside" or "outside" of a formal system, one could not easily define any relation between it and a system. Thus, it would seem that a "meta-proof", or a proof about proofs, is required. Specifically, one would desire to show conclusively that neither a proof nor a disproof of the proposition at hand is possible. In terms of the ideas presented in the previous paragraph, this is to say that, once the proper relation between terms has been established, no amount of manipulation will ever yield an equality or contradiction.

As far as beauty in proof is concerned, it is not difficult to agree on many of the characteristics a good proof should have; difficulty arises, however, when inquiries as to why such characteristics are important are made. Hardy and Avigad both speak of the importance of applicability in other areas of mathematics. This is taken to mean that a beautiful proof should have a sort of "rippling effect" and be representative of a more general truth whose power can be recognized on multiple levels. Thus, constructive proofs are typically preferred over nonconstructive proofs. Viewed in this light, such a proof serves a twofold purpose: to prove the proposition in question and also to open new avenues into separate branches of mathematics. It may be said, therefore, that one aspect of a beautiful proof is its ability to "interconnect" mathematics by revealing a certain level of similarity among otherwise disparate fields. More elusive, however, are qualities such as *elegance*, which might be defined by succinctness, clarity, and certainly those characteristics which Hardy mentions. Whether or not elegance is of any practical significance is uncertain. It seems, rather, to have aesthetic value, appealing to human notions of what is beautiful. Nonetheless, it is reasonable to suppose that precisely this impression of beauty may inspire a greater appreciation and understanding in the reader, thereby communicating the proof's message in a superior way. From a human standpoint, elegance is certainly a desirable trait for this reason.

It has been shown that the ability to prove propositions is an essential skill in furthering mathematical knowledge. Additionally, new perspectives on what proof represents and how to approach proof on a universal level, as well as what constitutes beauty in proof, have been given. An explicit enumeration of all possible propositions relating to a formal system being impossible[8], it is imperative that methods exist by which direct, formal derivations may be circumvented. This is the power of proof—a means by which to assert the truth or falseness of human conjectures, ultimately leading to greater understanding.

I pledge my honor that I have abided by the Stevens Honor System.

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[8] A Cantor diagonal argument shows this to be true. See, for example, Hofstadter, Douglas. *Gödel, Escher, Bach*, pp. 418-30.

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