Prove that if $0 < x < \frac{\pi}{2}$, then:

$$\sec^6 x + \csc^6 x + \sec^6 x \csc^6 x \geq 80$$

The answer is on the next page.

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First notice that the $0 < x < \frac{\pi}{2}$ stipulation is only there to make sure that all the trigonometric expressions are defined. (Secant is not defined at $\frac{\pi}{2}$ and cosecant is not defined at 0.) Actually, by the symmetry of these functions the result is good for any $x \neq 0, \frac{\pi}{2}$.

Now let's work with those expressions. By using algebra and trigonometric identities we can reduce the trigonometric expression to:

$$\begin{aligned}
\sec^6 x + \csc^6 x + \sec^6 x \csc^6 x &= \frac{1}{\cos^6 x} + \frac{1}{\sin^6 x} + \frac{1}{\cos^6 x}\frac{1}{\sin^6 x} \\
&= \frac{(\sin^2 x)^3 + (\cos^2 x)^3 + 1}{\cos^6 x \sin^6 x} \\
&= \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \cos^2 x \sin^2 x + \cos^4 x) + 1}{\cos^6 x \sin^6 x} \\
&= \frac{\sin^4 x - \cos^2 x \sin^2 x + \cos^4 x + 1}{\cos^6 x \sin^6 x} \\
&= \frac{(\sin^2 x + \cos^2 x)^2 - 3\sin^2 x \cos^2 x + 1}{\cos^6 x \sin^6 x} \\
&= \frac{2 - 3\sin^2 x \cos^2 x}{\cos^6 x \sin^6 x}
\end{aligned}$$

To determine how the above behaves, we use a change of variable. Let $u = \sin x \cos x$. What we wish to show is now:

$$\frac{2 - 3u^2}{u^6} \geq 80$$

or, getting everything on one side and factoring:

$$\left(u + \frac{1}{2}\right)\left(u - \frac{1}{2}\right)(80u^4 + 20u^2 + 8) \geq 0$$

Now $80u^4 + 20u^2 + 8$ is always positive (it's the sum of squares), so what we really need to show is that

$$\left(u + \frac{1}{2}\right)\left(u - \frac{1}{2}\right) \geq 0$$

or

$$u^2 \geq \frac{1}{4}$$

But we know that $u = \sin x \cos x$. Putting that back in and solving for it reduces this to

$$-1 \leq 2\sin x \cos x \leq 1$$

or

$$-1 \leq \sin 2x \leq 1$$

which is obviously always true.