Problem of the Week
Oct. 8, 2003

Find all integral values n **for which the polynomial** $f(n) = n^4 - 4n^3 + 10n^2 - 12n + 86$ **is a perfect square.**

It does not take too much playing around with alternative ways to write $f(n)$ to stumble upon

$$\begin{aligned} f(n) &= n^4 - 4n^3 + 10n^2 - 12n + 86 \\ &= (n^2 - 2n + 3)^2 + 77 \end{aligned}$$

Thus after setting $f(n)$ equal to k^2 where k is an integer we find that equivalently we need to solve

$$\begin{aligned} 77 &= k^2 - (n^2 - 2n + 3)^2 \\ &= \big(k - (n^2 - 2n + 3)\big)\big(k + (n^2 - 2n + 3)\big) \end{aligned}$$

Since both expressions on the right are integers, this boils down to setting the two factors of the right hand side equal to various integer factorizations of 77 that only involve two numbers. This is made a little easier by the fact that every integer factorization of 77 has two numbers. There are eight integer factorizations to check, namely $(\pm 1, \pm 77)$, $(\pm 77, \pm 1)$, $(\pm 7, \pm 11)$ and $(\pm 11, \pm 7)$. As an example, for our first pair $(1, 77)$ we must check the system

$$\begin{aligned} k - (n^2 - 2n + 3) &= 1 \\ k + (n^2 - 2n + 3) &= 77 \end{aligned}$$

and see if it has a solution in integers. This first example in fact does, since solving it gives us the condition

$$n^2 - 2n - 35 = 0 \quad \Rightarrow \quad n = -5, 7$$

As it turns out there is only one other pair that works, that being $(7, 11)$ which reduces to

$$n^2 - 2n + 1 = 0 \quad \Rightarrow \quad n = 1$$

and those $n = 1, -5, 7$ are our answers.

This problem originally appeared in the last Princeton University Press mathematics book catalog. It was one of three problems which constituted a contest: solve the three problems and send them in and get your name put into a drawing for $250 worth of free books. They did make it sound like the contest would be a regular event...?!