Prove that there are infinitely many triples of positive integers a, b, c such that the greatest common divisor of a, b and c is 1, and the sum $a^2b^2 + b^2c^2 + a^2c^2$ is the square of an integer.

The answer is on the next page.

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This problem is easiest if we first start off with a guess as to the relationship between a, b and c. Since we know they must be relatively prime, let's let a be odd, $b = 2$, and let $a + b = a + 2 = c$. (So c is just the next odd number after a.) We are now guaranteed that a, b, c are relatively prime, and there are obviously an infinite number of possibilities here. Furthermore:

$$
\begin{aligned}
a^2b^2 + b^2c^2 + a^2c^2 &= 4a^2 + a^2c^2 + 4c^2 \\
&= 4a^2 + a^2(a+2)^2 + 4(a+2)^2 \\
&= a^4 + 4a^3 + 12a^2 + 16a + 16
\end{aligned}
$$

It is not immediately obvious, but a little effort (or a quick check with a program like Scientific Notebook or Maple) reveals that the above

$$a^4 + 4a^3 + 12a^2 + 16a + 16 = (a^2 + 2a + 4)^2$$

and we're done, since $a^2 + 2a + 4$ is an integer. Any triple $(a, 2, a + 2)$ where a is odd will guarantee relative primeness and the relationship desired.

Note: The stipulations that $b = 2$ and a be odd were convenient but they can actually be relaxed, and so the following is true. Any relatively prime triple (a, b, c) where $a + b = c$ will produce a perfect square in our expression. The algebra is only slightly messier—try it!