Problem of the Week
Oct. 29, 2003

Two trick-or-treaters, Amy and Brandon, decide to trick-or-treat on opposite sides of the same street. Each house has a 50% chance of giving a child candy. If Amy's side of the street has 10 houses and Brandon's side has 11 houses, what is the probability that Brandon bags more candy than Amy? (We are assuming each house gives out the same amount of candy.)

Brandon wins 50% of the time. (Before you think this means Brandon has no advantage, remember that ties are certainly possible and thus Amy wins *less* than half of the time.) There are at least three different ways to show this.

(1) The first is by direct calculation. Let X and Y be the random variables which count how many houses give Amy and Brandon candy respectively. Note that X and Y are binomial random variables, and so we have

$$\begin{aligned} P(X<Y) &= \sum_{i=0}^{10}\left(P(X=i)\sum_{j=i+1}^{11}P(Y=j|X=i)\right) \\ &= \sum_{i=0}^{10}\left(\binom{10}{i}(.5)^{10}\sum_{j=i+1}^{11}\binom{11}{j}(.5)^{11}\right) \\ &= (.5)^{21}\sum_{i=0}^{10}\sum_{j=i+1}^{11}\binom{10}{i}\binom{11}{j} \end{aligned}$$

At this point a program like Scientific Notebook or Maple can confirm that

$$\sum_{i=0}^{10}\sum_{j=i+1}^{11}\binom{10}{i}\binom{11}{j} = 2^{20}$$

(You can also consult various identities involving binomial coefficients for the general result.) We are left with $P(X<Y)=0.5$.

(2) We can do the problem more easily by noting some symmetries that are available. Before Brandon visits the "extra" house on his side of the street, after he and Amy have both visited 10 houses each, the probability that he has more candy than Amy is exactly the same as the probability that Amy has more candy than he does. Call this probability p. This means the probability that the two kids are tied after 10 houses each is $1-2p$. Now there are only two ways Brandon can win: (a) he is ahead after 10 houses each and it doesn't matter what the extra house does, or (b) he is tied with Amy but the extra house gives him candy. By what we said before, the probability of (a) is p and the probability of (b) is $(1-2p)\frac{1}{2}$. Thus the probability Brandon wins is $p+(1-2p)\frac{1}{2}=\frac{1}{2}$.

(3) Probably the slickest way to show the probability Brandon wins is 50% is to construct a bijection between the number of ways he can win and the number of ways he doesn't. This is not as "out of nowhere" as you might think since we are dealing with successes and failures, wins and losses, in other words opposites.

Pretend we have all possible outcomes laid out before us. Essentially each member of this list is a set of twenty-one 1's and 0's, the first 10 entries denoting Amy's successes and failures and the last 11 entries denoting Brandon's. Each of the entries that is a win for Brandon has more 1's in the last 11 entries than in the first 10. Our bijection involves simply "flipping" each 1 to a 0 and vice-versa.

We need to check that the flip operation is one-to-one and onto. It's certainly one-to-one, since we can't flip two different sequences and get the same sequence. Showing that it is onto involves showing that any non-win can be flipped into a win, and this too is true: outcomes where Brandon tied with Amy become an outcome where Brandon wins by one house, and outcomes which were losses for Brandon get flipped into wins. Thus our "flip" operation is a bijection and there is one win for Brandon for every non-win for Brandon, and thus these sets must be the same size.

One final note. Now that you've seen the arguments, it is not too difficult to see that the answer is 50% for any number of houses, so long as Brandon gets to visit one more house than Amy. Try it for yourself (whichever way you wish) and see!

This problem was adapted from a problem in *Challenging Mathematical Problems with Elementary Solutions, Vol. 1* by A.M. Yaglom and I.M. Yaglom.