

The following three problems were presented all at once.

**What is the 1,000,000,000th decimal digit of  $\alpha = \sum_{n=1}^{\infty} (2 \cdot 10^{n^2})^{-n}$ ? And the 999,999,999th? And the 1,000,000,001st?**

The answer is on the next page.

**What is the 1,000,000,000th decimal digit of  $\alpha = \sum_{n=1}^{\infty} (2 \cdot 10^{n^2})^{-n}$ ? And the 999,999,999th? And the 1,000,000,001st?**

Multiply the top and bottom of the unit fraction involved by  $1 = \frac{5^n}{5^n}$ . You get:

$$\begin{aligned}\alpha &= \sum_{n=1}^{\infty} \left( \frac{1}{2^n \cdot 10^{n^3}} \right) \cdot \frac{5^n}{5^n} \\ &= \sum_{n=1}^{\infty} \left( \frac{5^n}{10^n \cdot 10^{n^3}} \right) \\ &= \sum_{n=1}^{\infty} \frac{5^n}{10^{n+n^3}}\end{aligned}$$

and this equivalent expression is in nice neat decimal form. Since the numerator of the fraction is increasingly much smaller than the denominator, we really don't need to worry about the summation—our decimal is essentially a bunch of  $5^n$  expressions separated by lots of zeros. (Compute the first few terms of this sum and you'll see this in action pretty quickly.)

So we can now concentrate on finding what  $n$  makes  $n + n^3 = 1,000,000,000$  or thereabouts. A simple calculation shows that  $n = 999$  comes up a bit short at 997,003,998 and  $n = 1000$  is the first  $n$  that eclipses it at 1,000,001,000. This means that our  $5^{1000}$  figure starts at the 1,000,001,000th decimal place and extends left, with a bunch of zeros separating its first digit from the last digit of  $5^{999}$ , which starts at the 997,003,998th decimal place. But  $5^{1000}$  has far fewer than 1000 digits (it has 233), and so it terminates long before reaching the 1,000,000,000th decimal place. All three of the digits asked for are zeros.

A hunter walks 10 miles south, 10 miles east, and 10 miles north only to find himself in the same place as he started. What kind of animal is he hunting?

In other words, describe all points (that is, starting points) on the surface of the earth (which we assume to be a perfect sphere) for which the above can happen.

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This is a famous problem. I won't provide a rigorous proof, but there are two types of places. The first is the north pole: after the first (south) leg, you are 10 miles south of the pole and walking east does not change that; it only rotates you around the pole.

The second set of places requires recognizing that very near the south pole you can "circle the earth" by walking a very short distance. In other words, if I am 1 foot from the south pole then the circle of latitude around the earth there is approximately  $2\pi$  feet long, and if I walk 10 miles east from there I will traverse that circle about  $\frac{10mi}{2\pi ft}$  times. Thus if I find a circle of latitude that is exactly 10 miles long, then *any* point 10 miles north of that will land me back where I started after going 10 miles south, 10 miles east, and 10 miles north: the first leg will place me on that circle of latitude, the second leg I will "circle the globe" exactly once, and thus the last leg I will retrace my steps from the first leg.

The latitude I chose is not the only one, however, as any latitude length that divides evenly into 10 miles will do the trick as well. Thus If I am 10 miles north of a latitude that is exactly 5 miles long, I will still end up right back where I started after my trip, only on this trip I will have circled the latitude twice. Similar phenomena occur for latitudes exactly  $\frac{10}{3}$  miles long, 2.5 miles long, 2 miles long, 1 mile long, 0.5 miles long, etc. In general we arrive back where we started if we start 10 miles north of a latitude that is exactly  $\frac{10}{k}$  miles long where  $k$  is a positive integer.

Two circles of radius  $r$  are centered at  $(-a, 0)$  and  $(a, 0)$  in the Cartesian plane. (We assume that  $a, r > 0$ .) Take a point  $P$  on the first circle, a point  $Q$  on the second circle, and call  $M$  the midpoint of the segment  $\overline{PQ}$ . What is the set described by  $M$  as  $P$  and  $Q$  vary on the two circles?

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Let's move to the complex plane for this one, so now a point  $(x, y)$  can be thought of as  $z = x + yi$  where  $i = \sqrt{-1}$ . In the complex plane any circle centered at  $z_0$  with radius  $r$  can be written as  $C = z_0 + re^{\theta i}$ , where  $\theta$  is simply the angle corresponding to the point at that angle from  $z_0$  on the circle. Our two circles are thus

$$\begin{aligned} C_P &= -a + re^{\theta i} \\ C_Q &= a + re^{\alpha i} \end{aligned}$$

Fix  $P$  and  $Q$  for a minute so we can see what one of our points looks like. This basically means fix  $\theta$  and  $\alpha$  for a minute. Then  $M$  is:

$$\begin{aligned} M &= \frac{P + Q}{2} \\ &= \frac{-a + re^{\theta i} + a + re^{\alpha i}}{2} \\ &= \frac{r}{2} (e^{\theta i} + e^{\alpha i}) \end{aligned}$$

So now we have a description of our set: they are all of the points that fit the above description of  $M$ . The only things that vary in the above description are the angles  $\theta$  and  $\alpha$ , so the set we require is the set of all  $M$  matching the above description such that  $0 \leq \theta, \alpha < 2\pi$ .

Great, so what does the set look like? To see this, fix  $\theta$  for a minute. Our  $M$  now looks like  $M = v + \frac{r}{2}e^{\alpha i}$  where  $v = \frac{r}{2}e^{\theta i}$  is point of that is  $\frac{r}{2}$  away from the origin at angle  $\theta$ . This is a circle whose diameter goes from the origin to point  $2v$ , which means the diameter is  $r$ . As  $\theta$  varies this simply rotates the center of the same circle (and thus the circle itself) around to the various  $\theta$  between 0 to  $2\pi$ . The figure swept out is a circle of radius  $r$  centered at 0. And that is our set.