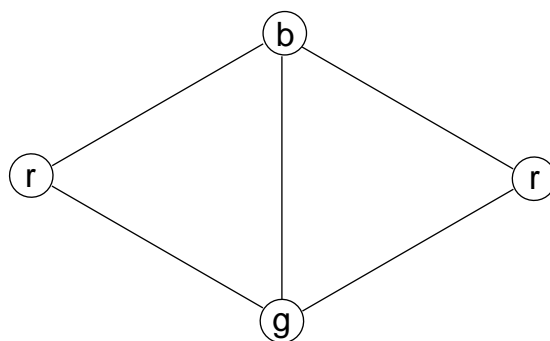


**Two questions this week:**

- 1. If every point on the plane is painted red, green, or blue, show that there must exist two points of the same color exactly one inch apart.**
- 2. Find a way to color the plane with nine colors such that no two points one inch apart are the same color.**

(1) We'll show this by assuming the opposite is true (namely that every two points one inch apart are different colors) and showing that it leads to a contradiction. Construct an equilateral triangle whose sides are one inch. By assumption, the corners of this triangle must all be different colors. Pick any side, say the side whose colors are blue and green, and form another equilateral triangle like so:



Since no points an inch apart are the same color, the remaining corner of that triangle is forced to be red. Now the two red points shown are  $\sqrt{3}$  inches apart, and in fact the above figure demonstrates that *any* point  $\sqrt{3}$  inches away from that original red must in fact be red. (Why? Because for any point  $\sqrt{3}$  inches from the original red we can form the above figure.) But if every point  $\sqrt{3}$  inches from the original red point is red as well, then there is in fact an entire red circle of radius  $\sqrt{3}$  inches centered at the original red point. But this is a contradiction, since any red circle with radius larger than  $\frac{1}{2}$  inch must have an infinite number of points on it distance 1 inch apart.

I'll mention in passing that the above argument works for any unit (not just one inch) and for any color (not just red), and so we have in fact shown that, given any distance  $d$  and any one of the three colors, then there are always two points  $d$  apart that are that color.

(2) Tile the plane in the following fashion. (The numbers one through nine refer to the colors.)

1	2	3
4	5	6
7	8	9

As long as the side  $s$  of each square is between  $\frac{1}{2} < s < \frac{\sqrt{2}}{2}$ , you are guaranteed no points one inch apart are the same color. Why? Since the side length  $s$  is less than  $\frac{\sqrt{2}}{2}$  you are guaranteed that no points within any square are one inch apart. And since side length  $s$  is larger than  $\frac{1}{2}$ , then any two squares of the same color are more than one inch apart.

This problem was taken from the 1988 Putnam Competition.