If $a+b+c=0$, then what does the following expression equal?

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)$$

As you may have figured out just by plugging in some suitable numbers, if $a+b+c=0$ then the above expression (surprisingly) always comes out to be 9. Here's why.

Let's look at the left-hand parenthetical expression first. Getting a common denominator and factoring gives us:

$$\begin{aligned}
\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c} &= \frac{b^2c-bc^2+ac^2-a^2c+a^2b-ab^2}{abc} \\
&= \frac{c^2(a-b)+ab(a-b)-(ac+bc)(a-b)}{abc} \\
&= \frac{(a-b)(c^2+ab-ac-bc)}{abc} \\
&= \frac{(a-b)[c(c-a)-b(c-a)]}{abc} \\
&= -\frac{(a-b)(b-c)(c-a)}{abc}
\end{aligned}$$

This is a handy fact to know, because it makes simplifying the right-hand parenthetical expression a little easier as well. Why? To make things easier to see, let's make the substitutions

$$\begin{aligned}
X &= b-c \\
Y &= c-a \\
Z &= a-b
\end{aligned}$$

Now take a close look at $Y-Z$, $Z-X$, and $X-Y$. For $Y-Z$ for instance we have

$$Y-Z=b+c-2a=-3a$$

where to get the last equality we used the fact that $a+b+c=0$. Doing the same for our other capital letters shows that $Z-X=-3b$ and $X-Y=-3c$. Now we're in business because we can solve for a, b, and c and substitute into our right-hand parenthetical expression to get

$$\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=-\frac{1}{3}\left[\frac{Y-Z}{X}+\frac{Z-X}{Y}+\frac{X-Y}{Z}\right]$$

That expression on the right should look familiar. We've simplified that before when we simplified the left-hand parenthetical expression (just with different letters). So we now know

$$\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b} = -\frac{1}{3}\left[-\frac{(X-Y)(Y-Z)(Z-X)}{XYZ}\right]$$

$$= \frac{1}{3}\frac{(-3a)(-3b)(-3c)}{(b-c)(c-a)(a-b)}$$
$$= -9\frac{abc}{(a-b)(b-c)(c-a)}$$

And now combining our two simplified expressions gives the answer 9.