

**One solution of the equation  $a^2 + b^2 + c^2 + 2 = abc$  is  $a = 4, b = 3,$  and  $c = 3$ . Do there exist integer solutions of this equation with  $a, b$  and  $c$  all larger than 10? Either find such a solution or show that none exists.**

Note that, since  $a, b$  and  $c$  are all symmetric in this problem, any solution we get for them we can permute to get another solution. For instance, since  $a = 4, b = 3,$  and  $c = 3$  is a solution, so is  $a = 3, b = 3,$  and  $c = 4$ . Surprisingly enough, this turns out to be one of the keys to the solution! Let's see why.

If we think of the equation given in terms of only one of the variables, the equation is really just a quadratic equation. We'll pick  $c$  to focus on, and we get

$$c^2 - abc + (a^2 + b^2 + 2) = 0$$

And this is really the other key to the solution: Once  $a$  and  $b$  are plugged in and this is a quadratic equation in  $c$ , then there are really *two* possible values for  $c$ !. (Quadratic equations have *two* roots.) By the quadratic formula

$$c = \frac{ab \pm \sqrt{(ab)^2 - 4(a^2 + b^2 + 2)}}{2}$$

For  $a = 4$  and  $b = 3$  the quadratic formula gives us our possible values  $c = 3$  (which we knew) and  $c = 9$ . So we know another triple that works is  $a = 4, b = 3$  and  $c = 9$ . But because of the symmetry, this means that  $a = 9, b = 4$  and  $c = 3$  is also a solution. Now plugging in this new  $a$  and  $b$  and solving our quadratic equation for  $c$  gives us new possible values of  $c = 3, c = 33$ . Permute the large values to  $a$  and  $b$  and keep solving for  $c$  until you get big numbers. Starting with our original values you get the chain:

$$\begin{aligned} (4, 3, 3) &\rightarrow (9, 4, 3) \\ &\rightarrow (33, 9, 4) \\ &\rightarrow (293, 33, 9) \\ &\rightarrow (9660, 293, 33) \end{aligned}$$

and that last answer fits the bill for our question.