**Suppose that the positive real numbers $a_1, a_2, \ldots, a_{100}$ satisfy**

$$\begin{aligned} a_1 \geq a_2 \geq \ldots \geq a_{100} &\geq 0 \\ a_1 + a_2 &\leq 100 \\ a_3 + a_4 + \ldots + a_{100} &\leq 100 \end{aligned}$$

**Determine the maximum possible value of $a_1^2 + a_2^2 + \ldots + a_{100}^2$, and find all possible sequences of $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.**

Let's let $a_1^2 + a_2^2 + \ldots + a_{100}^2 = S$ so we don't have to write the whole expression all the time. It should be fairly obvious that the $a_i$ that maximize $S$ will, in fact, satisfy the conditions

$$\begin{aligned} a_1 + a_2 &= 100 \\ a_3 + a_4 + \ldots + a_{100} &= 100 \end{aligned}$$

since if we come up with a "solution" that doesn't fit the above conditions we can just increase one of the $a_i$ until the above conditions are met, and $S$ will only get bigger. In particular, note that $a_1 + a_2 + \ldots + a_{100} = 200$ should be satisfied. Let's look for such a solution. Using the above two facts and the combination of them I just mentioned we can rewrite the above as

$$\begin{aligned} S &= (100 - a_2)^2 + a_2^2 + \ldots + a_{100}^2 \\ &= 100^2 - 200a_2 + 2a_2^2 + a_3^2 + \ldots + a_{100}^2 \\ &= 100^2 - (a_1 + a_2 + \ldots + a_{100})a_2 + 2a_2^2 + a_3^2 + \ldots + a_{100}^2 \\ &= 100^2 + (a_2^2 - a_1a_2) + (a_3^2 - a_3a_2) + \ldots + (a_{100}^2 - a_{100}a_2) \\ &= 100^2 + (a_2 - a_1)a_2 + (a_3 - a_2)a_3 + \ldots + (a_{100} - a_2)a_{100} \end{aligned}$$

Since our $a_i$ sequence is non-increasing, all of the expressions in parentheses are either zero or negative, and we know that the maximum $S$ can be is $100^2$ or 10000. $S$ can achieve 10000 if and only if all of the parenthetical expressions are zero. So we are looking for a solution where

$$(a_i - a_2)a_i = 0$$

for all $i \neq 2$. This can only happen when the $a_1 = a_2 = \ldots = a_k$ for some $k$ and $a_{k+1} = \ldots a_{100} = 0$ for all the rest.

So running through the various possibilities for $k$ gives us our solutions. We must still satisfy the original inequalities imposed on us, and it turns out that these limit the optimal solutions to two. If $k = 1$ then $a_2 = a_3 = \ldots = 0$ and so we obviously have $a_1 = 100$, and this achieves the maximum. Now whenever $k \geq 2$ we should have $a_1 = a_2 = 50$ in order to maximize the first condition. Letting $a_1 = a_2 = a_3 = a_4 = 50$ and setting all the rest to zero maximizes the second inequality as well and is the only other sequence that achieves the maximum we're looking for.