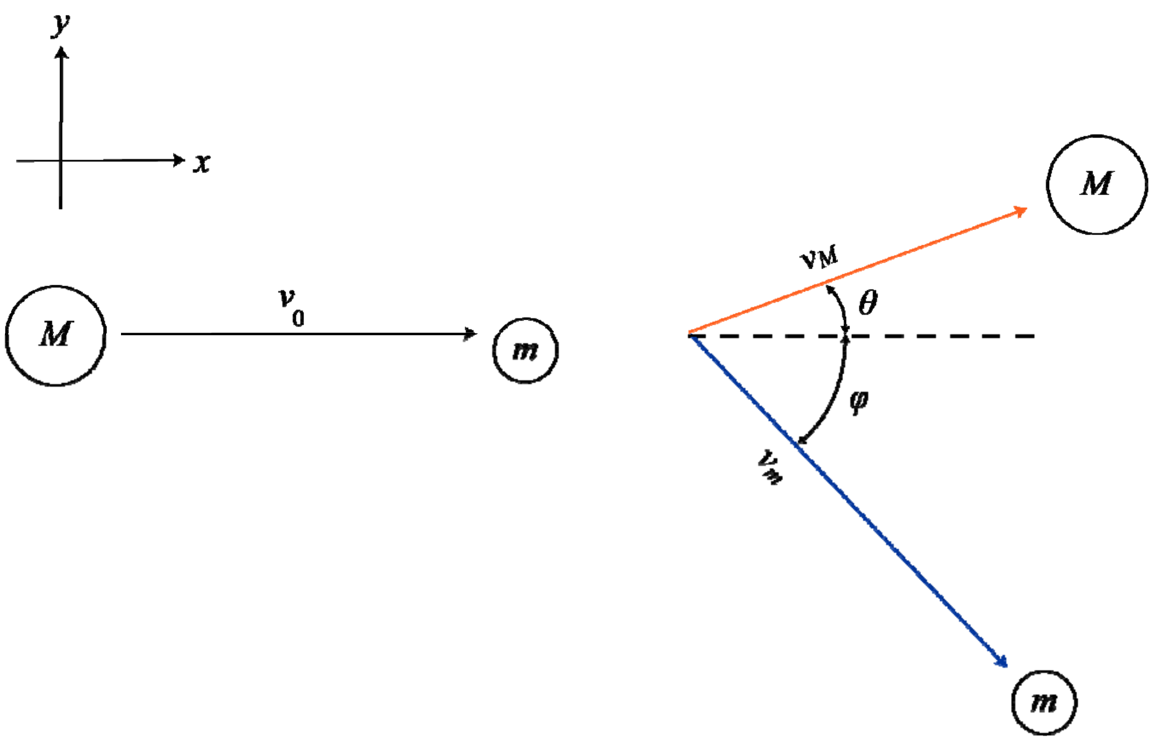
Setting up a system of equations

Suppose ball M has a velocity of v_0 before the collision and a velocity of v_M after the collision, and that ball m has a velocity of v_m after the collision.



The collision will result with conservation of momentum in both the x and y dimensions and kinetic energy. Thusly:

$$\sum p_{xi} = \sum p_{xf} \rightarrow Mv_0 = Mv_M \cos\theta + mv_m \cos\varphi$$

$$\sum p_{yi} = \sum p_{yf} \rightarrow 0 = Mv_M \sin\theta - mv_m \sin\varphi$$

$$\sum K_i = \sum K_f \rightarrow \frac{1}{2}Mv_0^2 = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2$$

Then, we can write:

$$v_0 = v_M \cos\theta + v_m \frac{m}{M}\cos\varphi$$

$$v_M \sin\theta = v_m \frac{m}{M}\sin\varphi$$

$$v_0^2 = v_M^2 + v_m^2 \frac{m}{M}$$

Solving for v_M and v_m in terms of v_0

Taking the second equation, we find: $v_M = v_m \frac{m}{M} \sin\varphi \csc\theta$

Then, substituting this into the first: $v_0 = \left(v_m \frac{m}{M} \sin\varphi \csc\theta \right) \cos\theta + v_m \frac{m}{M} \cos\varphi$

We can rewrite this as: $v_0 = v_m \frac{m}{M} \sin(\theta + \varphi) \csc\theta \rightarrow v_m = v_0 \frac{M}{m} \csc(\theta + \varphi) \sin\theta$

Now, to find v_M in terms of v_0 by substituting for v_m:

$$v_M = \left(v_0 \frac{M}{m} \csc(\theta + \varphi) \sin\theta \right) \frac{m}{M} \sin\varphi \csc\theta \rightarrow v_M = v_0 \csc(\theta + \varphi) \sin\varphi$$

Finally, substitute these two into the energy equation:

$$v_0^2 = \left(v_0 \csc(\theta + \varphi) \sin\varphi \right)^2 + \left(v_0 \frac{M}{m} \csc(\theta + \varphi) \sin\theta \right)^2 \frac{m}{M}$$

Solving for θ in terms of φ

We can factor out v_0^2 and manipulate to come out with: $\sin^2(\theta + \varphi) = \sin^2\varphi + \frac{M}{m} \sin^2\theta$

Again rewriting:

$$\sin^2\theta \cos^2\varphi + 2\sin\theta \cos\theta \sin\varphi \cos\varphi + \cos^2\theta \sin^2\varphi = \sin^2\varphi + \frac{M}{m} \sin^2\theta$$

Using some more trigonometric relations, we can get to:

$$\sin^2\theta \cos 2\varphi + \sin\theta \cos\theta \sin 2\varphi = \frac{M}{m} \sin^2\theta$$

We can factor out a $\sin\theta$ and divide by $\cos\theta$, which will result in:

$$\tan\theta\left(\frac{M}{m}-\cos 2\varphi\right)=\sin 2\varphi$$

And, finally: $\theta=\tan^{-1}\left(\frac{\sin 2\varphi}{M/m-\cos 2\varphi}\right)$

Finding the maximum value

To find the maximum value, we'll find the critical point:

$$\frac{\partial\theta}{\partial\varphi}=\frac{2[(M/m)\cos 2\varphi-1]}{1+(M/m)^2-(M/m)\cos 2\varphi}=0$$

For this to be true, then $\cos 2\varphi=\frac{m}{M}$ and $\sin 2\varphi=\sqrt{1-\left(\frac{m}{M}\right)^2}$

So: $\theta_{\max}=\tan^{-1}\left(\frac{\sqrt{1-(m/M)^2}}{M/m-m/M}\right)$ → $\theta_{\max}=\tan^{-1}\left(\frac{m/M}{\sqrt{1-(m/M)^2}}\right)$

We can write this more elegantly as

$$\theta_{\max}=\sin^{-1}\left(\frac{m}{M}\right)$$