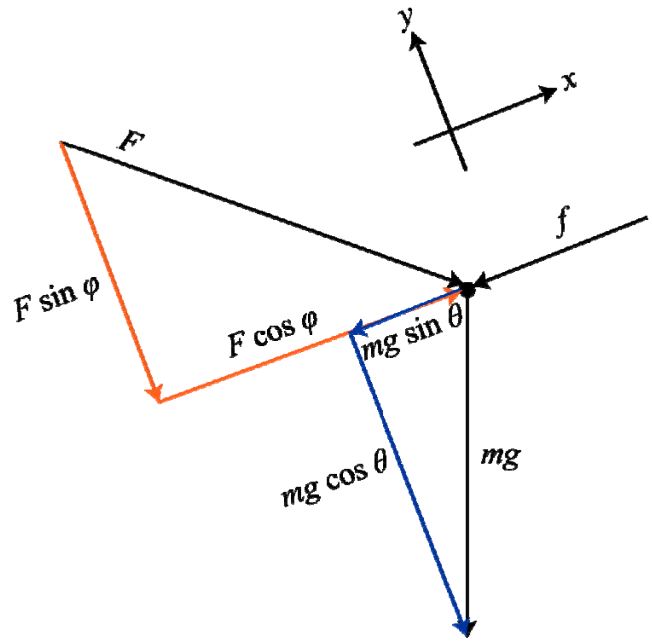
Finding a general equation for the coefficient of static friction



Consider a *xy*-coordinate system with the *x*-axis perpendicular to the plane.

Here, we have:

$$\sum F_x = N - mg\cos\theta - F\sin\varphi = 0$$
$$\therefore N = mg\cos\theta + F\sin\varphi$$
$$\therefore f = \mu_s N = \mu_s\left(mg\cos\theta + F\sin\varphi\right)$$

The friction force *f* can act in either direction depending on the applied force *F*.

$$\sum F_y = F\cos\varphi - mg\sin\theta \pm f = 0$$

$$F\cos\varphi - mg\sin\theta \pm \mu_s\left(mg\cos\theta + F\sin\varphi\right) = 0$$

$$\mu_s = \pm\frac{mg\sin\theta - F\cos\varphi}{mg\cos\theta + F\sin\varphi}$$

Now we must find the minimum value of μ_s regardless of *F*.

Finding the minimum value for small *F*

There is a critical value of *F* at $F_{crit} = mg\sin\theta\sec\varphi$ where $\mu_s = 0$. If *F* is chosen below this value, then *f* will act to prevent the block from sliding down the plane, hence:

$$\mu_s = \frac{mg\sin\theta - F\cos\varphi}{mg\cos\theta + F\sin\varphi}$$

The minimum value for μ_s for any *F* in this range will be the maximum of this function for $0 \le F < F_{crit}$.

$$\frac{\partial\mu_s}{\partial F} = -\frac{mg\cos(\varphi-\theta)}{\left(mg\cos\theta + F\sin\varphi\right)^2}$$

The derivative is negative on [0, F_{crit}) and therefore the maximum value will occur at $F = 0$.

$$\mu_s\big|_{F=0} = \tan\theta$$

Finding the minimum value for large *F*

Now, we must check when F is on (F_{crit}, ∞), where f will act to prevent the block from sliding up the plane, hence:

$$\mu_s = \frac{F\cos\varphi - mg\sin\theta}{F\sin\varphi + mg\cos\theta}$$

$$\frac{\partial\mu_s}{\partial F} = \frac{mg\cos(\varphi-\theta)}{(mg\cos\theta + F\sin\varphi)^2}$$

The derivative is always positive on (F_{crit}, ∞), and therefore the maximum value will be as $F \to \infty$.

$$\mu_s = \lim_{F\to\infty}\frac{F\cos\varphi - mg\sin\theta}{F\sin\varphi + mg\cos\theta}$$

$$\mu_s = \cot\varphi$$

Finding the overall values

So, we have two possibilities, $\mu_s = \tan\theta$ or $\mu_s = \cot\varphi$. Which one applies depends simply on which is bigger, so we must find the criteria for that.

If $\tan\theta \le \cot\varphi$, then it can be shown that $0^\circ < \theta + \varphi \le 90^\circ$

Oppositely, if $\tan\theta > \cot\varphi$ then $90^\circ < \theta + \varphi < 180^\circ$

This gives us the final result*:

$$\mu_s = \begin{cases} \cot\varphi, & 0^\circ < \theta + \varphi \le 90^\circ \\ \tan\theta, & 90^\circ < \theta + \varphi < 180^\circ \end{cases}$$

*We can note that for the special case $\theta + \varphi = 90^\circ$, $\tan\theta = \cot\varphi$.