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An Analysis of a Local Second-Moment Conserving Quasi-Lagrangian Scheme for Solving the Advection Equation

CHRISTOPHER L. KERR

Water Resources Program, Department of Civil Engineering, Princeton University, Princeton, New Jersey 08540

AND

ALAN F. BLUMBERG

Geophysical Fluid Dynamics Program, Princeton University, Princeton, New Jersey 08540

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I. INTRODUCTION

A local second-moment conserving quasi-lagrangian scheme developed by Egan and Mahoney [2] and extended by Pedersen and Prahm [4] for solving the advection equation is analyzed. This investigation establishes the ability of the scheme to advect a one-dimensional wedge distribution at a uniform velocity by comparing characteristics of the analytic distribution with those obtained through numerical advection. A comparison is also made between the scheme and several commonly implemented finite difference advection schemes. The one-dimensional wedge distribution was selected for investigation as several previous investigators have used it for examining the characteristics of various advection schemes. The characteristics of the local second-moment conserving quasi-lagrangian scheme have been investigated by Christensen and Prahm [1] and Pepper and Long [5].

II. GENERAL DESCRIPTION OF THE SCHEME

The numerical scheme utilizes a Eulerian rectangular grid over any domain. Within each rectangle, called a macro-cell, one separate, rectangular micro-cell is established based upon the constituent distribution within the macro-cell. The macro-cell, with an arbitrary constituent distribution $c(\xi, \eta)$ is replaced with a micro-cell of uniform rectangular constituent distribution with the same zeroth, first and second central moments as the macro-cell. The quantities (ξ, η) are non-dimensional coordinates

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with respect to the center of the macro-cell where $-0.5 \leqslant \xi \leqslant 0.5$ and $-0.5 \leqslant \eta \leqslant 0.5$. The micro-cell moments are obtained from

$$\bar{c} = \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} c(\xi, \eta)\, d\xi\, d\eta \tag{1}$$

$$m_x = \frac{1}{\bar{c}} \int_{-0.5}^{0.5} c(\xi, \eta)\, \xi\, d\xi \tag{2}$$

$$m_y = \frac{1}{\bar{c}} \int_{-0.5}^{0.5} c(\xi, \eta)\, \eta\, d\eta \tag{3}$$

$$\mu_x^2 = \frac{12}{\bar{c}} \int_{-0.5}^{0.5} c(\xi, \eta)(\xi - m_x)^2\, d\xi \tag{4}$$

$$\mu_y^2 = \frac{12}{\bar{c}} \int_{-0.5}^{0.5} c(\xi, \eta)(\eta - m_y)^2\, d\eta \tag{5}$$

and establish the micro-cell's amount of constituent, center of gravity and widths within the macro-cell. (See Fig. 1.)

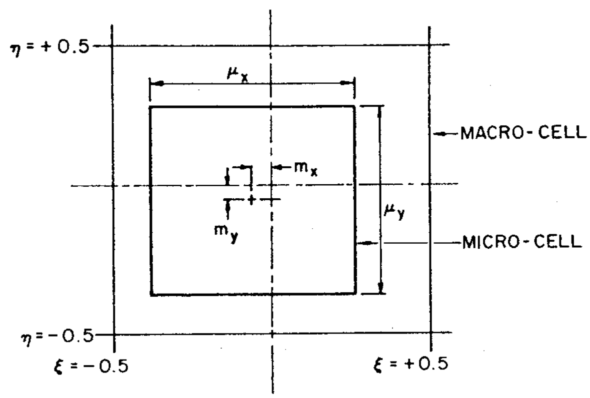


FIG. 1. The definition of a micro-cell within a macro-cell. The micro-cell cen m_y) is measured with respect to the macro-cell center of gravity, and the micro-cell widths are (μ_x, μ_y). Within the macro-cell, the (ξ, η) coordinates are used where $-0.5 \leqslant \xi \leqslant 0.5$ and $-0.5 \leqslant \eta \leqslant 0.5$.

At each time step the new position of an advected constituent is established by linearly translating each micro-cell, according to the prescribed velocity field, to a new location within the Eulerian grid. As each micro-cell is translated, information on the contribution it will make to the distribution of constituent within each macro-cell is recorded. When all micro-cells have been translated, the new constituent distribution within each macro-cell is used in conjuction with Eqs. (1) through (5) to establish new micro-cell details.

The scheme is quasi-lagrangian as at each time step micro-cells are translated with respect to the fixed grid and then immediately decomposed to the Eulerian macro-cell grid. At a local macro-cell level, the scheme conserves the individual macro-cell's constituent distributions zeroth, first and second central moments. However, for the global constituent distribution the zeroth moment is conserved while higher moments are modified.

III. Analysis of the Scheme

The numerical investigation follows the analysis procedure of Mahlman and Sinclair [3] to examine the ability of the scheme to solve the one-dimensional linear advection equation

$$\frac{\partial c}{\partial t} = -u \frac{\partial c}{\partial x} \tag{6}$$

where $c = c(x, t)$ is a constituent being advected at a constant velocity u along the x axis. The initial conditions imposed upon c are

$$c = \begin{cases} 0 & \text{for} \quad x \leqslant 5 \quad \text{or} \quad x \geqslant 5 \\ 1 + \dfrac{x}{5} & \text{for} \quad -5 \leqslant x \leqslant 0 \\ 1 - \dfrac{x}{5} & \text{for} \quad 0 \leqslant x \leqslant 5 \end{cases} \tag{7}$$

Through non-dimensionalizing the terms in Eq. (6) the only relevant parameter is the CFL criterion, σ, where $\sigma = u(\Delta t/\Delta x)$. Here Δx is the grid spacing and Δt is the time interval. The values of σ were chosen to equal 0.3125 and 0.03125 and were selected as each allows for investigation of advection of an integer number of wedge distributions with $\sigma = 0.3125$ corresponding to a typical choice in studies while $\sigma = 0.03125$ is sufficiently small to suppress space and time truncation errors.

An investigation of the behavior of the numerical scheme applied to Eq. (6) with the initial constituent distribution described by (7) can be performed by considering the following domain integrals:

$$\int c^m \, dx \simeq \sum_{i=1}^{n} (c)^m \qquad m = 1, 2, 4 \tag{8}$$

$$\int \left(\frac{\partial c}{\partial x}\right)^2 dx \simeq \sum_{i=1}^{n} (c_{i+1} - c_i)^2 \tag{9}$$

$$\int \left(\frac{\partial^2 c}{\partial x^2}\right)^2 dx \simeq \sum_{i=1}^{n} (c_{i+1} - 2c_i + c_{i-1})^2 \tag{10}$$

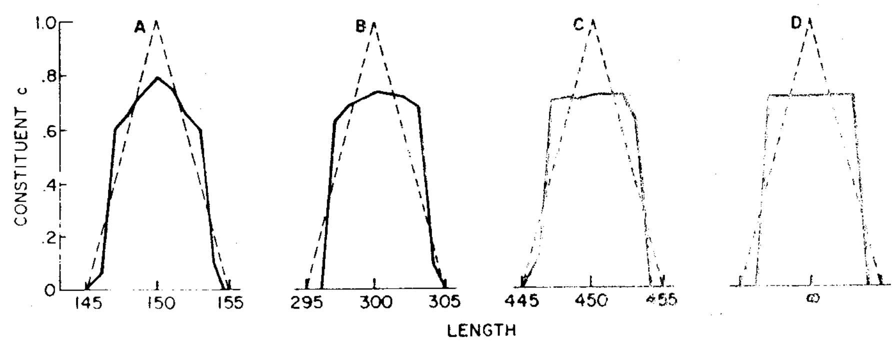
where n corresponds to the number of equally spaced grid points.

These time invariant integrals describe important characteristics of the constituent distribution. The integral (8) with $m = 1$ expresses conservation of the advected constituent. All schemes considered herein conserve this integral and therefore the integral has not been included in further discussions. The integral (8) with $m = 2$ is, in many physical applications, proportional to some form of energy, and, hence, the integral measures diffusive effects of the numerical scheme. The integral (8) with $m = 4$ is an expression of how the advected constituent distribution is able to maintain maximum values. The integral (9) measures the growth or decay of local constituent gradients, while (10) measures the growth or decay of local curvature of the constituent.

The constituent distributions obtained from advecting an initial wedge distribution fifteen, thirty, forty-five and infinite wedge-widths by the local second moment conserving quasi-lagrangian scheme with $\sigma = 0.3125$ are illustrated in Fig. 2. The constituent distributions in Fig. 2 show no presence of any phase difference between the numerical and analytic distibutions. Therefore, the computational velocity of the scheme equals the true analytic velocity. The numerical distributions also show no evidence of either negative values or non-physical disturbances leading or trailing the distribution.

All the integrals, (8) through (10) of the numerically advected distribution show low correlation with respect to their initial values of a wedge distribution. These changes in the numerically advected distribution integrals are associated with the decomposition of the original wedge distribution into a rectangular one, which is illustrated at infinity wedge-widths in Fig. 2. The integrals of the numerically advected distribution are, therefore, all bounded as, at infinity wedge-widths; they all reach steady state values.

The auto-spectra of c^2 obtained from the cases described in Fig. 2 are shown in Fig. 3a. The figure illustrates that the scheme preserves low wave numbers while



	INITIAL CONDITIONS	A 15 WIDTHS	B 30 WIDTHS	C 45 WIDTHS	D ∞ WIDTHS
$\int c^2 dx$	= 3.4000	3.3490	3.4470	3.4480	3.5287
$\int c^4 dx$	= 2.1330	1.6440	1.7010	1.7020	1.7788
$\int (\frac{\partial c}{\partial x})^2 dx$	= 0.4000	0.5597	0.7710	0.7640	1.0082
$\int (\frac{\partial^2 c}{\partial x^2})^2 dx$	= 0.2400	0.7853	1.3320	1.2730	2.0164

FIG. 2. A comparison between the analytical and numerical distributions for advection intervals of 15, 30, 45 and ∞ wedge-widths with $\sigma = 0.3125$.

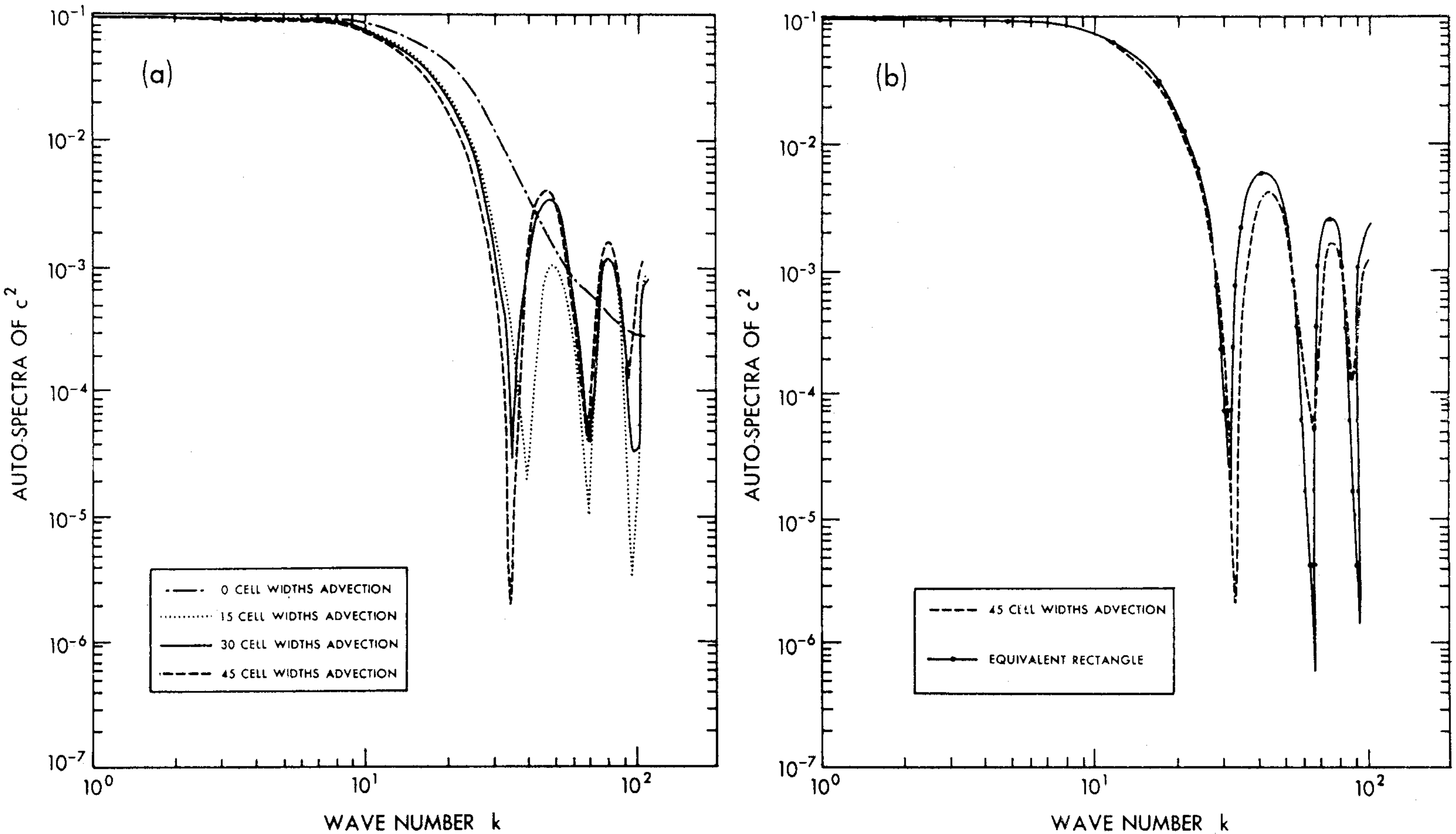


FIG. 3. The auto-spectra of c^2 for (a) the initial wedge distribution and the numerical distributions for advection intervals of 15, 30 and 45 wedge-widths and (b) a rectangular distribution and the numerical distribution after an advection interval of 45 wedge-widths with $\sigma = 0.3125$.

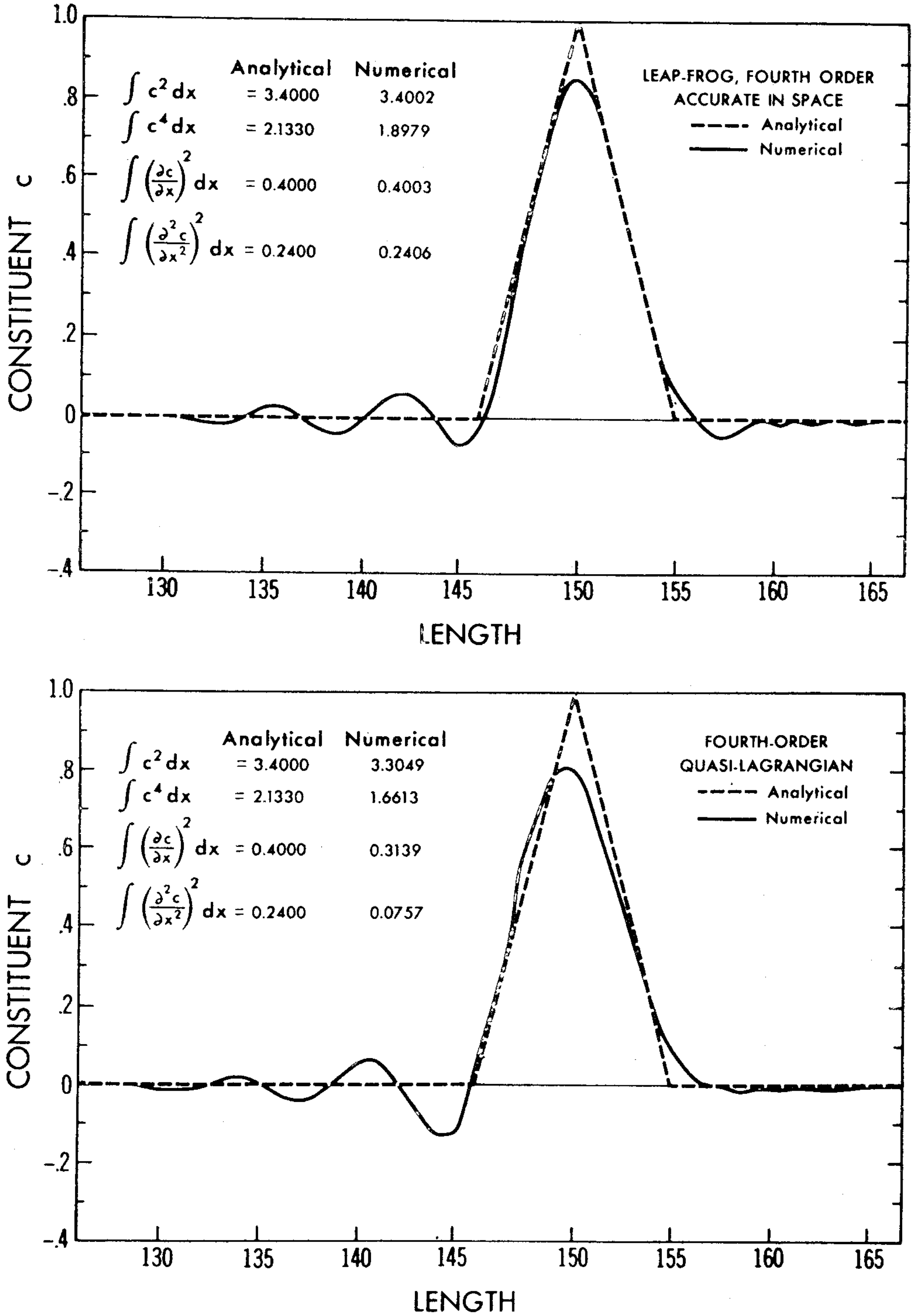
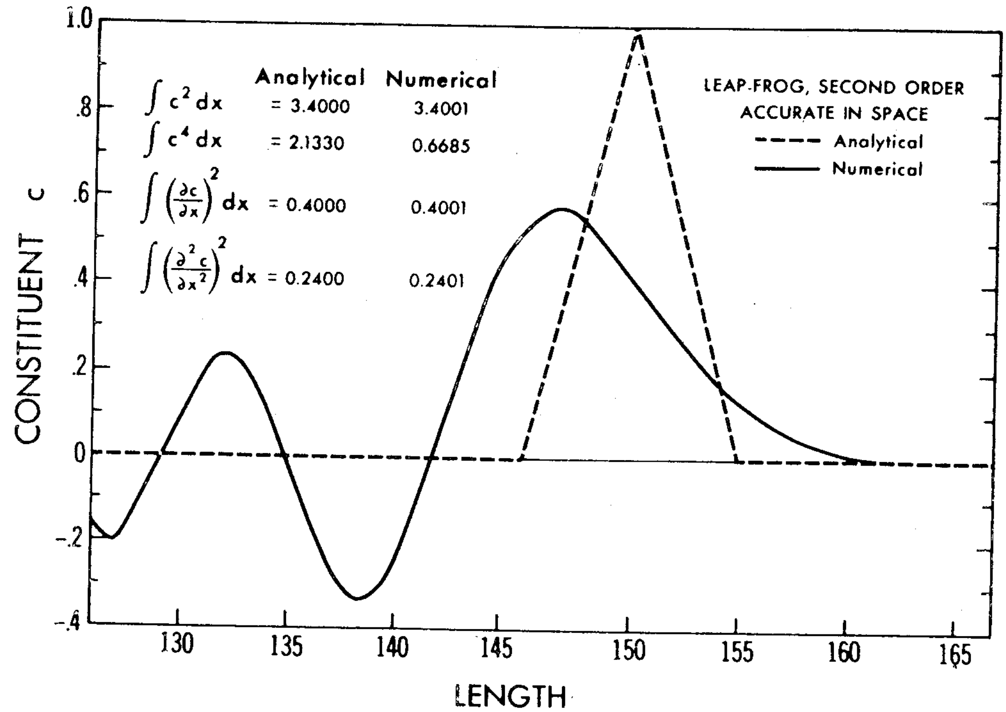


FIG. 4. A comparison between the analytical distribution and several numerically obtained distributions after an advection interval of 15 wedge-widths with $\sigma = 0.3125$.



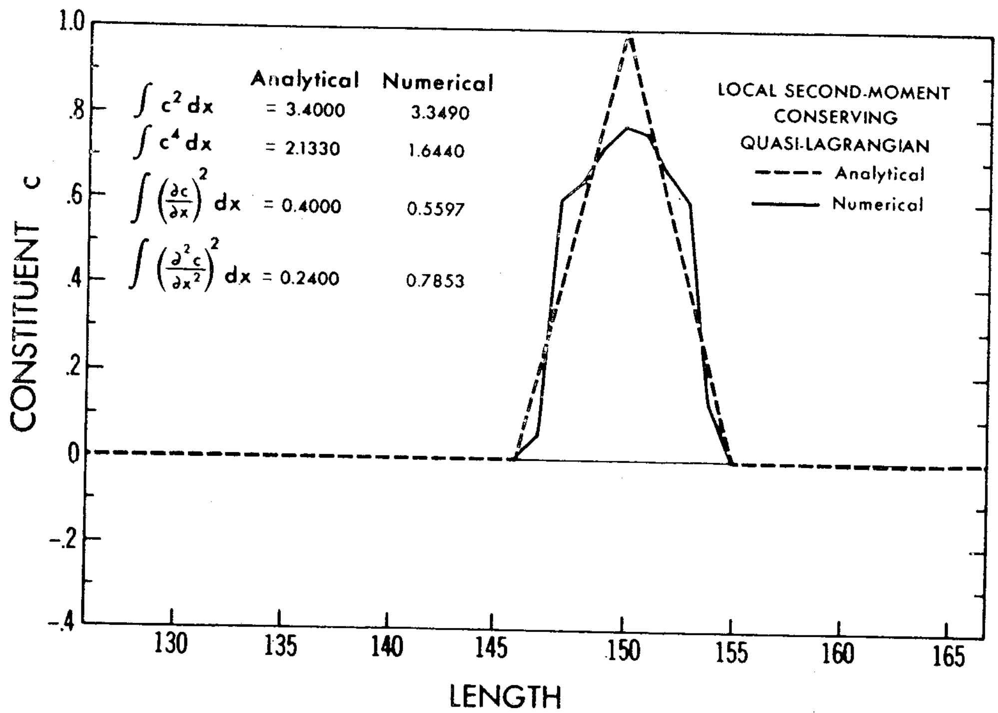


FIG. 4—Continued.

amplifying certain high wave numbers. The amplification of wave numbers is associated with the decomposition of the initial wedge into a rectangular distribution as illustrated in Fig. 3b. This amplification of certain high wave numbers may lead to instability in non-linear self advection ($u(\partial u/\partial x)$) problems.

The time truncation error of the local second-moment conserving quasi-lagrangian scheme is studied by propagating the initial wedge distribution fifteen wedge-widths for σ equal to 0.3125 and 0.03125. With $\sigma = 0.03125$ all the features exhibited by the constituent distributions with $\sigma = 0.3125$ were evident. At the reduced value of σ, however, the wedge decomposition to a rectangular distribution was more evident.

To evaluate the local second-moment conserving quasi-lagrangian scheme in context with other schemes, equivalent numerical studies with an initial wedge distribution advected fifteen wedge-widths with $\sigma = 0.3125$ are made with several commonly implemented finite difference approximations of the advection equation, (see Mahlman and Sinclair [3]). The finite difference schemes investigated include the leap frog in time, with second and fourth order accuracy in space schemes, and a quasi-lagrangian, Euler in time and fourth order accurate in space scheme.

An illustration of the characteristics of each scheme in advecting the initial wedge distribution is shown in Fig. 4. The leap frog, fourth order accurate in space scheme shows a small phase lag with the peak value suppressed. Non-physical disturbances are evident in the distribution and all integrals with the exception of $\int c^4\, dx$ are conserved. The scheme demonstrates significant improvement over the leap frog second order accurate in space scheme which produces significant phase lag and appreciable decrease in peak value. Prominent, non-physical disturbances are also evident behind the distribution with all integrals except $\int c^4\, dx$ being conserved. The quasi-lagrangian scheme demonstrates no evidence of phase lag, some presence of non-physical disturbances, a decrease in peak values, and a supression of all integrals.

IV. Conclusions

An investigation of a second-moment conserving quasi-lagrangian scheme demonstrated several unique features. The obvious advantages exhibited by the scheme which appeared in the constituent distribution include no possibility of negative distribution values appearing, no evidence of phase lag between the numerical and analytic distribution, no presence of non-physical disturbances either leading or trailing the distribution and that the higher the CFL criterion the more accurate the shape of the constituent distribution compared to the initial one. The scheme can also be implemented in regions containing irregular closed boundaries without any complications.

The disadvantages of the scheme are associated with the decomposing of the initial wedge constituent distribution into a rectangular one through advection. This introduced the effect of non-conserving the global integrals and amplifying certain high wave numbers in the autospectra of c^2 which may lead to instability in non-linear self advection problems. The advantage of a more accurate, constituent distribution being obtained with a higher CFL criterion can also be a disadvantage as a low CFL criterion may be required to accurately describe the prescribed velocity field.

The investigation of several commonly implemented finite difference advection schemes demonstrates the characteristics of these schemes within the study. A comparison of the schemes suggests that the local second-moment quasi-lagrangian scheme is competitive with these other schemes.

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References

1. O. Christensen and L. P. Prahm, *J. Appl. Meteorol.* **15** (1976), 1284–1294.
2. B. A. Egan and J. R. Mahoney, *J. Appl. Meteorol.* **11** (1972), 312–322.
3. J. D. Mahlman and R. W. Sinclair, *in* "Fate of Pollutants in the Air and Water Environments" (I. H. Suffet, Ed.), Vol. 8, Part 1, pp. 223–252. Wiley, New York, 1977.
4. L. B. Pedersen and L. P. Prahm, *Tellus* **26** (1974), 594–602.
5. D. W. Pepper and P. E. Long, *J. Appl. Meteorol.* **17** (1978), 228–233.

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