Discriminating completions of hyperbolic groups

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Abstract

In this paper we develop a notion of a completion G^A of a group G by an Abelian group A. The completion G^A is an A-operator group satisfying certain Lyndon's axioms. We prove that if G is a torsion-free hyperbolic group then G^A is G-discriminated by G. This implies various model theoretic and algorithmic results concerning G^A.

1 Introduction

1.1 A-groups and A-completions

Let A be an additively written abelian group. We term A *unitary* if it comes equipped with a distinguished non-zero element, which we denote by 1. Such unitary abelian groups can be likened to topological spaces with base points. Distinguishing a non-zero element in an additively written abelian group amounts simply to the specification of one of its cyclic subgroups, with a specific choice of generator. A typical example of a unitary abelian group is the additive group A^+ of a unitary ring A with the ring identity as the distinguished element. Unitary abelian groups form a category where morphisms are the unitary homomorphisms (i.e., homomorphisms which map 1 to 1) and the subobjects are unitary subgroups (subgroups containing the distinguished element 1). We will

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henceforth adopt the notation $A' \leq A$ in order to express the fact that A' is a unitary subgroup of the unitary abelian group A.

Now let A be an unitary abelian group. A group H is termed an A-group if it comes equipped with a function $H \times A \to H$:

$$(h, \alpha) \mapsto h^\alpha$$

satisfying the following conditions for arbitrary $g, h \in H$ and $\alpha, \beta \in A$:

$$h^1 = h, \quad h^{\alpha+\beta} = h^\alpha h^\beta,$$

$$g^{-1} h^\alpha g = (g^{-1} h g)^\alpha,$$

and if g and h commute,

$$(gh)^\alpha = g^\alpha h^\alpha.$$

In the event that A is a unitary ring we assume also that

$$h^{\alpha\beta} = (h^\alpha)^\beta.$$

We shall simply refer to such A-groups, in all of their incarnations, as *exponential* groups. Every group G is a $\mathbb{Z}$-group, where here the set $\mathbb{Z}$ of integers can be viewed either as a unitary abelian group with distinguished element 1 or as a unitary ring, with $\mathbb{Z}$ acting on G by exponentiation as usual. More generally, when A is a torsion–free, unitary abelian group, the assumption that G be an A-group, means not only that A acts on G so that the natural conditions involving exponentiation hold, but also that the infinite cyclic subgroup of A generated by 1 acts on G in precisely the way that $\mathbb{Z}$ acts on any group. In the event that A is a torsion–free, unitary abelian group we shall sometimes, without explicit mention, denote the subgroup of A generated by 1 by $\mathbb{Z}$.

The most well-known, non-trivial, examples of A-groups are those in which A is a binomial ring [11], the field $\mathbb{Q}$ of rational numbers [3] and the ring $\mathbb{Z}[x]$ of integral polynomials in a single variable x, introduced by R.C. Lyndon in [18]. Lyndon introduced such A-groups in order to describe the solutions of equations in a single variable with coefficients in a free group - we shall have more to say about this later. A detailed investigation of A-groups and A-completions (see the paragraph that follows for the definition) when A is a unitary ring, has recently been carried out by A. Myasnikov and V. Remeslennikov in [21] and [22]. As already noted, they termed such groups *exponential groups* and we have expanded their terminology to include the case of unitary abelian groups, as detailed above.

There is a connection between these two types of A-groups. In order to understand how this comes about, recall that the *centroid* ([17]) of a group H is the set $\Gamma(H)$ of all mappings $\gamma : H \longrightarrow H$ satisfying the following two conditions (here h^γ is the image of $h \in H$ under $\gamma \in \Gamma(H)$) :

$$(gh)^\gamma = g^\gamma h^\gamma \quad \text{if} \quad gh = hg$$

and
$$(g^{-1}hg)^{\gamma} = g^{-1}h^{\gamma}g \ .$$
It is easy to verify that Γ becomes a ring on defining multiplication to be composition of mappings and addition by
$$h^{\gamma+\kappa} = h^{\gamma}h^{\kappa}.$$

Notice that $\Gamma(H)$ is an unitary ring and H can be viewed as $\Gamma(H)$–group in the obvious way.

Now suppose that A is an unitary abelian group and that H is an A-group. There is a canonical homomorphism, say, $\rho : A \longrightarrow \Gamma(H)$, where $\rho(\alpha)$ is simply the exponentiation in H defined by $\alpha \in A$. Thus, a group H is an A–group if and only if there exists an unitary homomorphism of A into the centroid of H. This means that every A–group is also an R-group, where R is a unitary subring of $\Gamma(H)$ generated by $\rho(A)$. The structure of such unitary rings R generated by homomorphic images of A will play an important part in the understanding of so-called A–completions of various groups (see the remarks that follow).

For a unitary abelian group A, the appropriate class of exponential groups constitutes a category in which one can define the usual notions of morphism (A-homomorphism), subobject (A-subgroup) and quotient.

If A' is a unitary subgroup of a unitary abelian group A, then every A-group can be viewed also as an A'-group, via restriction. On the other hand, suppose that $A' \leq A$ and G is an A'-group. Then there exists an A-group H and an A'-homomorphism $\mu : G \longrightarrow H$ such that for every A-group K and every A'-morphism $\theta : G \longrightarrow K$, there exists a unique A-homomorphism $\phi : H \longrightarrow K$ such that $\phi\mu = \theta$, i.e. the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\mu"] \arrow[d, "\theta"'] & H \arrow[dl, "\phi"] \\ K & \end{tikzcd}$$

The A–group H is unique up to isomorphism of A–groups. It is termed the (A', A)*–completion* of G or the A*–completion* of G if A' is the subgroup of A (subring of A) generated by 1. Following [21], we denote the completion in every instance simply by G^A.

Let now A be a unitary abelian group. By general nonsense there exists a unitary ring, which we denote by $fr(A)$ and a unitary homomorphism $\alpha : A \longrightarrow fr(A)^+$, such that for every unitary ring R and every (unitary) homomorphism $\beta : A \longrightarrow R^+$, there exists a unique unitary ring homomorphism $\gamma : fr(A) \longrightarrow R$ such that $\gamma\alpha = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & fr(A) \arrow[dl, "\gamma"] \\ R & \end{tikzcd}$$

We term $fr(A)$ the *unitary ring freely generated by* A. The structure of $fr(A)$ turns out to be important here. In **2** we shall prove that $fr(A)$ can be embedded in a free associative algebra over the field $\mathbb{Q}$ of rational numbers, provided A is torsion-free. More precisely, suppose that $\{1\} \cup X$ is a maximal $\mathbb{Z}$-linearly independent set of elements of A. Let L_X be the subgroup of A generated by X and let $T(L_X)$ be the tensor power of L_X over $\mathbb{Q}$:

$$T(L_X) = \bigoplus_{n=0}^{\infty} T_n(L_X).$$

Let ϕ_X be the canonical, unitary embedding of A into $T_0(L_X) \oplus T_1(L_X)$ (see **2.3**). Then the following theorem holds.

Theorem A *Let A be a torsion–free, unitary abelian group. Then the canonical extension of ϕ_X to a unitary ring homomorphism from $fr(A)$ into $T(L_X)$ is a monomorphism.*

In **3** we shall prove that if G is a group and A is a unitary abelian group then

$$G^A = G^{fr(A)}.$$

Thus our A–completions (A is a unitary abelian group) coincide with $fr(A)$–completions ($fr(A)$ is a unitary ring), which means that we can avail ourselves of [22].

It is worth giving some examples here.

If $A = gp(1) \oplus C$, here C is an infinite cyclic group, then $fr(A) = \mathbb{Z}[x]$, where $\mathbb{Z}[x]$ is the ring of integral polynomials in a single variable x. Hence, $G^A = G^{\mathbb{Z}[x]}$; in particular, the A-completion F^A of a free group F is just Lyndon's completion $F^{\mathbb{Z}[x]}$.

Similarly, if $A = \mathbb{Q}^+ \oplus C$ is a unitary abelian group (here $\mathbb{Q}^+$ is the additive group of rational numbers and C is an infinite cyclic group) with the identity of $\mathbb{Q}$ the distinguished element in A, then $fr(A) = \mathbb{Q}[x]$ is the ring of polynomials in the single variable x with coefficients in $\mathbb{Q}$ and $G^A = G^{\mathbb{Q}[x]}$.

Finally, let A be a torsion-free unitary abelian group A of rank one (i.e., $\{1\}$ is a maximal linear independent subset of A). Denote by $\mathbb{Q}_A$ the subring of $\mathbb{Q}$ generated by 1 and the rational numbers $1/p$, for which the p-Sylow subgroup of the quotient $A/gp(1)$ is nontrivial. Then $fr(A) = \mathbb{Q}_A$ and $G^A = G^{\mathbb{Q}_A}$. Notice, that if $A = gp(1)$ then $\mathbb{Q}_A = \mathbb{Z}$, and in this event $G^A = G$.

In general, completions G^A are impossible to understand. However in the case where A is a torsion–free, unitary abelian group and G is a torsion–free CSA-group, the structure of both $fr(A)$ and $G^{fr(A)}$ can be reasonably well determined.

We recall from [22] that a group G is called a *CSA group* if every maximal abelian subgroup M of G is malnormal, i.e., $M^g \cap M = 1$ for every $g \in G - M$.

The class of CSA-groups is quite substantial. It includes all abelian groups, all torsion-free hyperbolic groups [21] and all groups acting freely on Λ-trees [1] as well as many one-relator groups – a complete description of one-relator CSA-groups was obtained by D. Gildenhuys, O. Kharlampovich and A.G. Myasnikov in [13].

We shall discuss this, as well as other results, in **2** and **3**. In particular, we shall prove the following theorem.

Theorem B *Let G be a torsion-free CSA-group and let $A \leq B$ be unitary rings of characteristic zero or torsion-free, unitary abelian groups. Then the canonical map $G^A \longrightarrow G^B$ is a monomorphism.*

We have focused attention here on $fr(A)$, the free unitary ring generated by A. In fact it turns out that there are a host of different unitary rings "generated" by A. We shall discuss this further in **2** and in **3**.

1.2 Extension of centralizers

The key to A-completions is a construction termed *extension of centralizers* by A. Myasnikov and V. Remeslennikov in [21], [22] and carefully investigated by them. We recall the definition. Let G be a group and let $C(u)$ denote the centralizer of an element $u \in G$. Suppose that $C(u)$ is abelian and that $\phi : \ C(u) \longrightarrow A$ is a monomorphism of $C(u)$ into an *abelian* group A. Then the amalgamated product

$$G(u, A) = \langle G * A \mid C(u) = C(u)^{\phi} \rangle$$

is called *an extension of the centralizer $C(u)$ by the group A with respect to ϕ* or, more briefly, *an A-extension of C(u)*. We shall, for the most part, assume that ϕ is simply inclusion, in which case we will omit any mention of it. It turns out [22], in the event that A is a unitary ring which is additively torsion–free, that A–completions of a torsion-free CSA group (in particularly, a torsion-free hyperbolic group) G can be constructed by repeatedly forming extensions of centralizers. Namely, the A-completion G^A is the union of an infinite chain of groups:

$$G = G_0 \leq G_1 \leq \ldots \leq G_\alpha \leq G_{\alpha+1} \leq \ldots \leq G_\delta = G^A;$$

here, for each ordinal $\alpha < \delta$ (writing C additively for the moment),

$$G_{\alpha+1} = \{G_\alpha * (C \otimes_{\mathbb{Z}} A) \mid \ c = c \otimes 1 \ \ (c \in C)\}$$

is an A-extension of some centralizer C in G_α and, for each limit ordinal λ, $G_\lambda = \cup_{\beta<\lambda} G_\alpha$.

In **5** we carry out a detailed investigation of residual properties of extensions of centralizers. This then allows us, making use of the construction above, to deduce some residual properties of A-completions of torsion-free hyperbolic groups.

1.3 G–groups, separation and discrimination

Let G be a given group. Then a group H is termed a G–group if it comes equipped with a monomorphism

$$\phi : G \longrightarrow H.$$

It follows that the class of G–groups is simply the class of those groups H which come equipped with a given embedding of G into H. We shall, for the most part, assume that ϕ is the inclusion of G into H. If G is the trivial group, then a G–group is simply a group. Notice that G is itself a G–group, where here ϕ is the identity map. Following usual custom, we term a homomorphism θ of a G–group H' into a G–group H a G–homomorphism if

$$\theta(g) = g \quad (g \in G).$$

We observe, on identifying G with its image in the G–group H, that a G–homomorphism from H to the G–group G is simply a *retraction* of H to G, i.e., a homomorphism of H to its subgroup G which is the identity on G. We will make much use of the two definitions that follow.

Definition 1 *Let H and K be G-groups and let$\mathcal{F}$ be a family of G-homomorphisms of H into K. We term $\mathcal{F}$ a G-separating family if for each $h \in H$, $h \neq 1$, there exists $\phi \in \mathcal{F}$ such that $\phi(h) \neq 1$. In this event we also say that K G-separates H.*

If $G = 1$, then we omit any reference to G and simply say that K separates H or H is separated by K. In this event, following usual practice, we also sometimes say that H is *residually* K.

Definition 2 *Let H and K be G–groups and let $\mathcal{F}$ a family of G–homomorphisms of H into K. We term $\mathcal{F}$ a G–discriminating family if for each finite subset H_0 of non-trivial elements of H, there exists a homomorphism $\phi \in \mathcal{F}$ such that $\phi(h) \neq 1$ for every $h \in H_0$ (we say that ϕ discriminates the finite set H_0). In this event we say that K G–discriminates H.*

Again, if $G = 1$ we say that K discriminates H; in this case some authors term H *fully residually* K or *ω–residually* K.

Surprisingly, the property of being discriminated by a group K is, in general, much stronger then that of being separated by K (see **1.5**). In fact it is not hard to obtain conditions which ensure that a group K which separates a second group H also discriminates H. We address this problem in [8]. Here we only mention the following result from **4**. If A is a torsion-free abelian group, then any group which is separated by A is discriminated by A, i.e., A discriminates everything it separates. A dual result holds for non-abelian CSA groups: if a non-abelian CSA group H is separated by a group K, then H is discriminated by K, i.e., H is discriminated by everything that separates it.

1.4 Residual properties of completions

Our primary objective in this paper is to prove that certain completions of torsion-free hyperbolic groups enjoy a number of residual properties. All of our results about residual properties of completions stem from a general technique ("the big powers method") involving relations of the form

$$h_1 {u_1}^{m_1} h_2 {u_2}^{m_2} \dots {u_n}^{m_n} h_{n+1} = 1,$$

where $h_1, \dots, h_{n+1}, u_1, \dots, u_n$ are elements of certain torsion-free groups and the m_i are integers. It turns out, in particular, that if the m_i are large enough in magnitude and if for each i, $h_{i+1}^{-1} u_i h_{i+1}$ does not commute with u_{i+1}, then such a relation cannot hold in any torsion-free hyperbolic group [24], [9]. This allows us to prove a number of residual properties of various completions of, in particular, torsion–free hyperbolic groups.

Theorem C1 *Let G be a torsion-free hyperbolic group and let $\mathbb{Z}[x]$ be the ring of integral polynomials in a single variable x. Then $G^{\mathbb{Z}[x]}$ is G-discriminated by G.*

The following definition underlines the difference between the completions $G^{\mathbb{Z}[x]}$ and $G^{\mathbb{Q}}$.

Definition 3 *Let A be a unitary subring of a unitary ring B of characteristic zero and let G be an A-group.*

1. *An (A, B)-completion of G is said to be of type L (after R. C. Lyndon [18]) if B^+/A^+ is torsion-free.*
2. *An (A, B)-completion of G is said to be of type B (after the first author [3]) if B^+/A^+ is a torsion group.*

This definition allows us to split the A-completion G^A of an arbitrary torsion-free hyperbolic group G by an unitary ring A of characteristic zero into two parts:

$$G \leq G^{\mathbb{Q}_A} \leq G^A, \tag{1}$$

where $\mathbb{Q}_A$ is the subring of A (and also that of $\mathbb{Q}$) consisting of the pre-images of the elements of A which are of finite additive order modulo Z, the subgroup generated by 1. Notice, that the group $A^+/{\mathbb{Q}_A}^+$ is torsion-free, and ${\mathbb{Q}_A}^+/Z$ is a torsion group. Hence $G^{\mathbb{Q}_A}$ is a $\mathbb{Q}_A$-completion of G of type B, and G^A is an A- completion of $G^{\mathbb{Q}_A}$ of type L. The decomposition (1) is termed the *B-L* *decomposition* of the completion G^A. For example, if A is a unitary algebra over $\mathbb{Q}$, then $\mathbb{Q}_A = \mathbb{Q}$ and therefore

$$G \leq G^{\mathbb{Q}} \leq G^A$$

is the the B-L decomposition of G^A; in particular, G^A, the $(\mathbb{Q}, A)$-completion of the $\mathbb{Q}$-group $G^{\mathbb{Q}}$, is of type L.

Completions of type L are similar, in many respects, to $\mathbb{Z}[x]$-completions.

Theorem C2 *Let G be a torsion-free hyperbolic group and let A be a unitary ring with free abelian additive group A^+. Then any A-completion G^A of type L of the group G (as $\mathbb{Z}$-group) is G-discriminated by G.*

Theorems C1 and C2 are based on Theorem 4 from Section **6** which is our main technical result about completions of hyperbolic groups. We will not formulate this result here. Instead we give one more of its applications to completions of hyperbolic groups by unitary abelian groups. Recall, that the subgroup $gp(1)$ is isolated in the unitary group A if the factor-group $A/gp(1)$ is torsion-free.

Theorem C3 *Let G be a torsion-free hyperbolic group and let A be a unitary abelian group. If A is free abelian and if $gp(1)$ is isolated in A, then G^A is G-discriminated by G.*

We shall give the proof of the main theorem, as well as its corollaries, in **6**.

1.5 Applications

One of the main applications of the theorems that we prove here is to what one might term *algebraic geometry over groups.* We refer the reader to the paper [7] where the basic theory is developed. In order to explain, let G be a given group, H a G–group, n a positive integer and $\mathbf{A}_n(H)$ the set of all n-tuples of elements of H:

$$\mathbf{A}_n(H) \ = \{(a_1, \ldots, a_n) \mid a_j \in H \ \} \ .$$

We denote the free product of G and the free group freely generated by $x_1, \ldots, x_n$ by $G[X]$. If S is a subset of $G[X]$, we define

$$V(S) \ = \{(a_1, \ldots, a_n) \in \ \mathbf{A}_n(H) \mid f(a_1, \ldots, a_n) = 1 \ \ \textit{for all} \ f \in S \ \} \ .$$

A subset Y of $\mathbf{A}_n(H)$ is termed *algebraic* if there exists a subset S of $G[X]$ such that $Y = V(S)$. We term two subsets S and S_0 of $G[X]$ *equivalent* if they define the same algebraic set, i.e., if

$$V(S) = V(S_0).$$

The G–group H is then called *equationally noetherian* if given any subset S of $G[X]$ there exists a finite subset S_0 of S which is equivalent to S. It turns out that if H is G-discriminated by G and if G is equationally noetherian, then so too is H [7]. This fact together with Theorems C2 and C3 implies the following result.

Theorem D1 *Let G be an equationally noetherian torsion-free hyperbolic group (in particular, a free group) and let A be a unitary ring (or free abelian unitary group) with free abelian additive group A^+. Then any A-completion G^A of type L of the group G (as $\mathbb{Z}$-group) is equationally noetherian.*

Another, perhaps the most important, application of the residual properties of completions comes out of a description of the irreducible components of algebraic sets. The discussion that follows will help to illuminate this remark. We refer the reader to [7] for more details of this and related topics and some appropriate references. Let G be an equationally noetherian torsion-free hyperbolic group. If we take the algebraic sets in $\mathbf{A}_n(G)$ as the basis for a topology on $\mathbf{A}_n(G)$, the so-called *Zariski topology*, then this topology is noetherian. Hence every closed set V of $\mathbf{A}_n(G)$ is a finite union of its irreducible components:

$$V = V_1 \cup \ldots \cup V_n.$$

Every algebraic set $V \subset \mathbf{A}_n(G)$ is uniquely determined by the following normal subgroup of $G[X]$:

$$I(V) = \{f(x_1, \ldots, x_n) \in G[X] \mid \ f(a_1, \ldots, a_n) = 1 \ \ for\ all\ (a_1, \ldots, a_n) \in V\ \}\ .$$

The factor-group $G[X]/I(V) = G_V$ is called the *coordinate group* of V. It turns out [7], that a finitely generated G-group H is the coordinate group of some irreducible algebraic set if and only if H is G-discriminated by G. This observation, together with Theorem C1 shows that all finitely generated G-subgroups of $G^{\mathbb{Z}[x]}$ are coordinate groups of irreducible algebraic sets. As far as we have been able to ascertain, given a torsion-free hyperbolic group G, all known examples of finitely generated G-groups which are G-discriminated by G, are G-embeddable in $G^{\mathbb{Z}[x]}$. It seems likely that $G^{\mathbb{Z}[x]}$ is the source of all examples of coordinate groups of irreducible algebraic sets over G.

For the next application of the main theorem we need some definitions from logic. A universal sentence in the language of group theory, with constants from G, is a formula of the type

$$\forall x_1 \ldots \forall x_n (\bigvee_{j=1}^{s} \bigwedge_{i=1}^{t} (u_{ji}(\bar{x}, \bar{g}_{ij}) = 1 \ \&\ w_{ij}(\bar{x}, \bar{f}_{ij}) \neq 1)$$

where $\bar{x} = (x_1, \ldots, x_n)$ are variables, $\bar{g}_{ij}$ and $\bar{f}_{ij}$ are arbitrary tuples of elements (constants) from G. We say that the G-groups H_1 and H_2 are *universally equivalent* if they satisfy exactly the same universal sentences with constants from G and we express this by writing $H_1 \equiv_\forall H_2$.

A subgroup G is called *existentially closed* in a group H, if any existential sentence with constants from G holds in the whole group H if and only if it holds in the subgroup G. Notice, that if G is existentially closed subgroup of a G-group H, then $G \equiv_\forall H$. It follows immediately from the definition that if a G-group H is G-discriminated by G, then G is existentially closed in H. Now it follows from Theorem C2 that

Theorem D2 *Let G be a torsion-free hyperbolic group. If A is a unitary ring of characteristic zero and G^A is an A-completion of G of type L, then G is existentially closed in G^A and $G^A \equiv_\forall G$.*

Finally, we want to mention one result on decidability of universal theories.

Theorem D3 *Let F be a non-abelian free group and let A be a unitary ring of characteristic zero. Then every A-completion F^A of type L has decidable universal theory.*

Indeed, by Theorem D2 F^A has the same universal theory as the free group F, which is decidable (G. Makanin [20]).

In conclusion, we would like to mention that this paper is a partially expanded version of the preprint [6].

2 Rings generated by unitary abelian groups

2.1 The free ring generated by a unitary abelian group

Let A be a unitary abelian group with distinguished element 1. We choose a presentation of A, qua abelian group, which includes a specific generator ξ_0 representing 1:

$$A =< \Xi; \mathcal{R} >_{ab} .$$

We have elected to add the subscript ab in the presentation symbol to emphasise the fact that this is a presentation in the category of abelian groups. We now define a ring $rg(A)$ by generators and defining relations as follows:

$$rg(A) =< \Xi; \mathcal{R} \cup \{\xi_0\xi - \xi, \xi\xi_0 - \xi \mid \xi \in \Xi\} > _{rg},$$

where now we use the subscript rg to emphasize the fact that we are working in the category of (associative) rings.

The following lemma is an easy consequence of the definition of $rg(A)$.

Lemma 1 *Let A be a unitary abelian group and let R be a unitary ring. Then every unitary homomorphism*

$$\phi : A \longrightarrow R^+$$

can be continued to a unique unitary ring homomorphism

$$\phi^* : rg(A) \longrightarrow R$$

.

Proof. Consider the image A' of A under ϕ in R. Then A' can be presented, qua abelian group, in the form

$$A' =< \Xi; \mathcal{R}' >_{ab},$$

where $\mathcal{R}'$ is a set of relators containing $\mathcal{R}$. Notice that the image of ξ_0 under ϕ is the unit element of R and that the subring of R generated by A' has a presentation on the generating set Ξ together with a system of relations which includes the defining relations of $rg(A)$. Hence ϕ can be continued to a unitary ring homomorphism of $rg(A)$ into R, as claimed.

It follows immediately from Lemma 1 and the very definition of $fr(A)$ that we have the

Corollary 1 *The ring $rg(A)$ generated by A coincides with $fr(A)$.*

We can think of $rg(A)$ as being defined in terms of a universal mapping property and therefore its definition is independent of the choice of generators Ξ and its presentation in terms of these generators. Notice that $fr(\mathbb{Z}^+) = \mathbb{Z}$ and $fr(\mathbb{Q}^+) = \mathbb{Q}$.

Lemma 2 *fr can be viewed as a covariant functor from the category of unitary abelian groups to the category of unitary rings.*

Proof. We observe to begin with, that if $\alpha : A \longrightarrow A'$ is a homomorphism of unitary abelian groups, then this gives rise, first to a unitary abelian group homomorphism of A into the additive group of the ring $fr(A')$ and thence, by Lemma 1, to a ring homomorphism $fr(\alpha) : fr(A) \longrightarrow fr(A')$. Since fr is clearly covariant, this completes the proof of the lemma.

Corollary 2 *If α is an isomorphism from A to A', then $fr(\alpha)$ is an isomorphism from $fr(A)$ to $fr(A')$.*

Corollary 2 follows immediately from Lemma 2.

2.2 Free associative algebras

Let A be an unitary abelian group and let α be the canonical homomorphism of A into $fr(A)$. Then it may well be the case that α is not a monomorphism. In the event that A is torsion–free, α is always a monomorphism. The proof, in the torsion–free case, that α is a monomorphism, carries with it some additional information about $fr(A)$, which will be needed in the sequel. In this subsection we prepare the way for a deeper understanding of $fr(A)$ by introducing some needed notation and proving some simple facts about free associative algebras.

To this end, let S be an unitary ring, let X a be a non–empty set and let $R = S < X >$ denote the free unitary ring over S freely generated by X. So

$$R = \bigoplus_{i=0}^{\infty} R_i,$$

where $R_0 = S$ and R_n is the free left S–module freely generated by all of the monomials in X of degree n. Two special cases will be of special interest to us - that in which $S = \mathbb{Z}$, the ring of integers, and the case where $S = \mathbb{Q}$, the field of rational numbers.

The following proposition is presumably well known, but we give a proof for completeness.

Proposition 1 *Let $R = \mathbb{Q} < X >$ be the free associative algebra over $\mathbb{Q}$ in the variables X and let $\{1\} \cup Y$ be a $\mathbb{Q}$-linearly independent subset of $R_0 \oplus R_1$. Then the unitary $\mathbb{Q}$–subalgebra of R generated by Y is a free associative unitary algebra over $\mathbb{Q}$ freely generated by Y.*

The proof of Proposition 1 requires a number of steps, which we record here as lemmas. The first of these is

Lemma 3 *Let $R = \mathbb{Q} < X >$ and let ϕ be a surjective unitary $\mathbb{Q}$–algebra homomorphism of R onto itself. If X is finite, then ϕ is an automorphism.*

Proof. Let

$$R(n) = \bigoplus_{i=n}^{\infty} R_i.$$

Then $R(n)$ is an ideal of R which is invariant under every endomorphism of R. Hence ϕ induces a homomorphism ϕ_n of the finite dimensional algebra $R/R(n)$ onto itself for every n. So ϕ_n is an automorphism. Since

$$\bigcap_{n=1}^{\infty} R(n) = 0$$

it follows that ϕ is an automorphism.

Corollary 3 *Let $R = \mathbb{Q}[X]$ and let $\{1\} \cup Y$ be a finite, linearly independent subset of $R_0 \oplus R_1$. If X is finite, then the unitary $\mathbb{Q}$–subalgebra of R generated by Y is a free $\mathbb{Q}$–algebra, freely generated by Y.*

Proof. We enlarge $\{1\} \cup Y$ to a basis $\{1\} \cup Y'$ of $R_0 \oplus R_1$. Then there is a map ϕ from X to Y' which induces an unitary $\mathbb{Q}$–algebra epimorphism, which we again denote by ϕ, from R to R. By Lemma 3, ϕ is an automorphism. Hence the unitary $\mathbb{Q}$–subalgebra of R generated by Y is a free $\mathbb{Q}$–algebra, freely generated by Y.

We are now in position to prove Proposition 1. To this end, let Y' be a finite subset of Y. We prove first that the unitary $\mathbb{Q}$–algebra S generated by Y' is free on Y'. To this end, observe that every element of Y' is a linear combination of elements in X. It follows that Y' is contained in the subspace spanned by X', where X' is a finite subset of X. Since the unitary $\mathbb{Q}$–subalgebra of R generated by X' is freely generated by X', it follows immediately from Corollary 3 that the unitary $\mathbb{Q}$–subalgebra of R generated by Y' is freely generated by Y'. It follows that the monomials in Y are linearly independent over $\mathbb{Q}$ and therefore that Y freely generates a free associative algebra over $\mathbb{Q}$, as required.

2.3 The ring fr(A)

We concern ourselves now with the case of a unitary abelian group A, where A is torsion–free. Let $\{1\} \cup X$ be a maximal $\mathbb{Z}$-linearly independent set of elements of A. Let L_X be the subgroup of A generated by X and let $T(L_X)$ be the tensor power of L_X over the field $\mathbb{Q}$ of rational numbers. So

$$T(L_X) = \bigoplus_{n=0}^{\infty} T_n(L_X)$$

is a graded algebra over $\mathbb{Q}$ which can be identified with the free associative unitary algebra with coefficients in $\mathbb{Q}$ generated by X. Here $T_0(L_X) = \mathbb{Q}$ and $T_1(L_X)$ is the subspace of $T(L_X)$ spanned by X. In particular, it follows that L_X is embedded in $T_1(L_X)$. We now identify the subgroup $gp(1)$ in A with the additive group of integers in $\mathbb{Q}$ by identifying 1, qua element of A, with 1, qua element of $\mathbb{Q}$. The embeddings of $gp(1)$ and L_X in $T(L_X)$ give rise to an unitary embedding ϕ_X of A into the additive abelian group of $T(L_X)$, viewed as a unitary abelian group; observe that the image of A is contained in $T_0(L_X) \oplus T_1(L_X)$. We denote by $fr_X(A)$ the unitary subring of $T(L_X)$ generated by $\phi_X(A)$. We will identify $fr_X(A)$ with $fr(A)$, which allows us to better understand the structure of $fr(A)$.

We denote below the *isolator* of a subgroup N of an abelian group A by $\sqrt{N}$; so by definition, $\sqrt{N}$ is the pre-image in A of the torsion subgroup of A/N.

When A is free abelian, $A/\sqrt{gp(1)}$ is also free abelian and hence we can express A as a direct sum $A = \sqrt{gp(1)} \oplus C$, where C is free abelian. Under these circumstances we choose X to be a free set of generators of C. Notice that $\sqrt{gp(1)}$ is infinite cyclic, generated by b, say; hence $mb = 1$, for a suitable choice of the integer m. We will make use of these remarks as well as the accompanying notation in the sequel.

Notice that it follows from the fact that A is embedded in $fr_X(A)$ that A is embedded in $fr(A)$. The following lemma then holds.

Lemma 4 *Suppose that A is free abelian and that X is chosen in the manner described above. Furthermore, suppose that ϕ is the extension of ϕ_X to a unitary ring homomorphism of $fr(A)$ onto $fr_X(A)$ and that Λ is the subring of $\mathbb{Q}$ generated by $1/m$. Then*

1. *ϕ is an isomorphism;*
2. *$fr(A) \cong \Lambda < X >$.*

Proof. Observe, that $A = \sqrt{gp(1)} \oplus L_X$. It follows, that $fr_X(A) = \Lambda < X >$. Now ϕ maps the subring Λ' of $fr(A)$ generated by $\sqrt{gp(1)}$ onto Λ. It follows then from the fact that $\sqrt{gp(1)}$ is embedded by ϕ in Λ, that ϕ maps Λ' isomorphically onto Λ. So, identifying Λ' with Λ, it follows that $fr(A)$ can be thought of as a unitary Λ–algebra generated by X. Since the image of $fr(A)$ under ϕ is $\Lambda < X >$, ϕ is an isomorphism from $fr(A)$ to $fr_X(A)$. We have therefore proved both of the assertions (1) and (2).

The following corollary is an immediate consequence of Lemma 4.

Corollary 4 *If A is a finitely generated unitary, free abelian group and if $gp(1)$ is isolated in A, then $fr(A)$ is a free associative unitary algebra over the ring $\mathbb{Z}$ of integers:*

$$fr(A) \cong \mathbb{Z} < X > .$$

We are now in a position to prove the following theorem, where we again adopt the notation developed above.

Theorem A *Let A be a torsion–free, unitary abelian group. Then the canonical extension of ϕ_X to a unitary ring homomorphism ϕ from $fr(A)$ onto $fr_X(A)$ is an isomorphism and hence $fr(A)$ is, additively, torsion–free.*

Proof. Let A' be a finitely generated, unitary subgroup of A. Let θ be the extension of the inclusion of A' into A to a unitary ring homomorphism of the unitary ring $fr(A')$ into $fr(A)$. Consider the composition $\psi = \phi\theta$ of $fr(A')$ into $fr_X(A)$. Now ϕ, restricted to A, is a monomorphism and ψ, restricted to A', is simply ϕ, restricted to A'. Let B' be the isolator of $gp(1)$ in A'. Then B' is cyclic, generated by, say b, and there exists an integer m such that $mb = 1$. Let $\{b\} \cup X'$ be a free set of generators of A'. Since ψ embeds A' into $fr_X(A)$, it follows that $\{b\} \cup X'$ is a linearly independent subset of $R_0 \oplus R_1$. Hence the unitary $\mathbb{Q}$-subalgebra of $R = \mathbb{Q} < X >$ generated by X' is, by Proposition 1, a free $\mathbb{Q}$–algebra, freely generated by X'. So if Λ is the subring of $\mathbb{Q}$ generated by B', it follows that the unitary subring of $fr_X(A)$ generated by $\psi(A')$ is isomorphic to $\Lambda < X' >$ and thence that ψ is an isomorphism. Since A' is an arbitrarily chosen unitary subgroup of A, it follows that ϕ is an isomorphism, as claimed.

The following corollaries are immediate consequences of Theorem A.

Corollary 5 *If $A = \mathbb{Q}^+ \oplus C$ is a unitary abelian group (here $\mathbb{Q}^+$ is again the additive group of rational numbers and C is an infinite cyclic group) with the identity of $\mathbb{Q}$ the distinguished element in A, then $fr(A) = \mathbb{Q}[x]$ is the ring of polynomials in the single variable x with coefficients in $\mathbb{Q}$.*

Corollary 6 *Let A be a torsion-free unitary abelian group A of rank one (i.e., $\{1\}$ is a maximal linearly independent subset of A). Denote by $\mathbb{Q}_A$ the subring of $\mathbb{Q}$ generated by 1 and the rational numbers $1/p$, for which the p-Sylow subgroup of the quotient $A/gp(1)$ is nontrivial. Then $fr(A) = \mathbb{Q}_A$.*

Corollary 7 *Let A be a torsion–free, unitary abelian group. If A is divisible, then $fr(A)$ is a free, unitary algebra over $\mathbb{Q}$, freely generated by X, where $\{1\} \cup X$ is a maximal linearly independent set in A.*

Corollary 8 *Let A be a torsion–free, unitary abelian group. Then $fr(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a free $\mathbb{Q}$-algebra, freely generated by X, where $\{1\} \cup X$ is a maximal linearly independent set in A. In particular, $fr(A) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq fr(A \otimes_{\mathbb{Z}} \mathbb{Q})$.*

Corollary 9 *Let $A \leq B$ be torsion-free, unitary abelian groups. Then the inclusion $A \longrightarrow B$ gives rise to the canonical embedding $fr(A) \longrightarrow fr(B)$.*

We need to extract some additional information from the proof of Theorem A.

Proposition 2 *Let $A \leq B$, where B is a torsion–free, unitary abelian group. Then the following hold.*

1. *If B is free abelian and B is A-discriminated by A, then $fr(B)^+$ is $fr(A)^+$-discriminated by $fr(A)^+$.*
2. *If A is divisible, then $fr(B)^+$ is $fr(A)^+$-discriminated by $fr(A)^+$.*
3. *If B is divisible, then $fr(B)^+$ is $fr(\sqrt{A})^+$-discriminated by $fr(\sqrt{A})^+$.*
4. *If B is finitely generated and A is isolated in B, then $fr(B)^+$ is $fr(A)^+$-discriminated by $fr(A)^+$.*
5. *If B is free abelian and $gp(1)$ is isolated in B, then $fr(B)^+$ is $gp(1)$-discriminated by $gp(1)$, i.e., $fr(B)^+$ is discriminated by $\mathbb{Z}$.*

Proof. We simply follow the steps in the proof of Theorem A in each of the situations described above.

1. To begin with we note that since B is A-discriminated by A, there is an A-homomorphism of the unitary abelian group B onto A, i.e., a retraction of B onto its subgroup A. It follows that A is a direct summand of B: $B = A \oplus A'$. Since A' is a subgroup of the free abelian group B, it too is free abelian and hence has a free basis Y', say. Choose a basis $\{b\} \cup Y$ for A, where here $b \in A$ is a generator of $\sqrt{gp(1)}$ in A. It follows that $mb = 1$ for some choice of the integer m. Put $X = Y \cup Y'$. It follows then from the proof of Theorem A, that if Λ is the subring of $\mathbb{Q}$ generated by $1/m$, then
$$fr(A) = \Lambda < Y > \quad and \quad fr(B) = \Lambda < X > .$$
Thus the additive group $fr(B)^+$ is a direct sum of $fr(A)^+$ and some abelian group C which is a free Λ- module. Obviously, C is separated by Λ^+ and hence C is separated by $fr(A)^+$. Now, by Lemma 7 (see Section **4.1**) the group C is discriminated by $fr(A)^+$, and consequently, $fr(B)^+ = fr(A)^+ \oplus C$ is $fr(A)^+$-discriminated by $fr(A)^+$.
2. Now a divisible subgroup of an abelian group is a direct summand. So, as in the proof of (1), we find that $B = A \oplus C$. In this case, however, $\sqrt{gp(1)}$ is an additive copy of $\mathbb{Q}$. So if $\{1\} \cup X$ is a maximal linearly independent set in A and Y a maximal linearly independent set in C, then $fr(A) = \mathbb{Q} < X >$ and $fr(B) = \mathbb{Q} < X \cup Y >$. An analogous argument to that given in (1), yields then the desired conclusion.
3. In order to prove (3), notice that $\sqrt{A}$ is a divisible subgroup of the divisible group B. Now the desired conclusion follows from (2).
4. Since here A is a direct summand of B, B is A–discriminated by A. So (4) is now an immediate consequence of (1).

5. Since $gp(1)$ is isolated in B, it is a direct summand. The desired conclusion follows readily either from (4) or (1).

There is an analogous ring $abr(A)$ to $fr(A)$, defined by adding to the relations of $fr(A)$ the extra relations $xy - yx$ where x and y range over Ξ (see **2.1**). There are then analogues of the results that we have proved for $fr(A)$. Indeed, all of these results carry over to $abr(A)$ where they take on a similar form with the various free algebras replaced by free commutative algebras, i.e., the usual polynomial algebras in the appropriate sets of variables. In fact if $\mathcal{V}$ is any variety of unitary (associative) algebras over $\mathbb{Q}$, then we can add to the defining relations of $fr(A)$ the relations that define the free algebras in $\mathcal{V}$.

We note here that it follows in much the same way as above, that if A is free abelian and unitary and if $gp(1)$ is isolated in A, then $abr(A) = \mathbb{Z}[X]$ is the ring of integral polynomials in an appropriately chosen set of indeterminates X. In general, $abr(A) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[X]$.

3 Completions

3.1 A-completions

We begin with the following lemma.

Lemma 5 *Let G be a group and let A be an unitary abelian group. Then $G^{fr(A)}$ is the A-completion of G.*

Proof. Let μ be the canonical homomorphism of G into $G^{fr(A)}$ and let θ be a homomorphism of G into an A-group H (see the diagram below). Consider the centroid $\Gamma(H)$ of H. Since H is an A-group, there is a unitary homomorphism ρ of A into the additive group of $\Gamma(H)^+$. So, by Lemma 1, ρ can be extended to an unique unitary homomorphism of $fr(A)$ into $\Gamma(H)$. So H can be viewed as an $fr(A)$-group. Therefore there is an unique $fr(A)$-homomorphism ϕ from $G^{fr(A)}$ into H such that $\phi\mu = \theta$.

$$\begin{tikzcd} G \arrow[r, "\mu"] \arrow[d, "\theta"'] & G^{fr(A)} \arrow[dl, "\phi"] \\ H & \end{tikzcd}$$

We can view ϕ as an A-homomorphism from the A-group $G^{fr(A)}$ into the A-group H. It follows that $G^{fr(A)}$ is the A-completion of G.

There is a relative version of the above lemma, which can be proved in exactly the same way. This is the content of the next lemma.

Lemma 6 *Let G be a group and let A be an unitary abelian group. Suppose that H is an A-group and that B is the unitary subring generated by the canonical image of A in the centroid $\Gamma(H)$. If B is commutative, then there is a unique A-homomorphism of $G^{abr(A)}$ into H.*

Given the hypothesis of Lemma 6, we will refer to $G^{abr(A)}$ as the *commutative A–completion* of G.

In the case where A is a torsion-free, unitary abelian group we have proved that $fr(A)$ is a subring of $\mathbb{Q} < X >$, the free unitary associative algebra, freely generated by an appropriately chosen set X (see **2.3**). As noted already in **2.3**, there is an analogous result for $abr(A)$, i.e., $abr(A)$ is a subring of $\mathbb{Q}[X]$, the free unitary commutative algebra, freely generated by an appropriately chosen set X, which is simply the algebra of polynomials over $\mathbb{Q}$ in the variables $x \in X$. These results are an important ingredient in determining the structure of the A–completions of a variety of different groups.

We remark, that there are a host of other rings in between $fr(A)$ and $abr(A)$, generated by a unitary abelian group A, as we noted at the end of **2.3**. We leave the obvious extensions of the present discussion to the interested reader.

3.2 Completions of CSA-groups

We have mentioned in **1.1** that, in general, completions G^A are impossible to understand. In the case where G is a torsion-free CSA-group and S is a unitary ring of characteristic zero a concrete description of G^S is obtained in [22].

By Lemma 5, the A-completion G^A of a group G by an unitary abelian group A is simply the $fr(A)$–completion $G^{fr(A)}$. In view of Theorem A, in the event that A is a torsion–free, unitary abelian group, $fr(A)$ is a unitary ring of characteristic zero. So the results obtained in [22] carry over in this case.

Combining results from [22] and **2** yields the following theorem.

Theorem B *Let G be a torsion-free CSA-group and let $A \leq B$ be unitary rings of characteristic zero (torsion-free, unitary abelian groups). Then the canonical map $G^A \longrightarrow G^B$ is a monomorphism.*

Proof. Assume that $A \leq B$ are unitary rings of characteristic zero. Let G^A be the A-completion of G. The group G^A can be viewed as a partial B-group (i.e., exponentiation by B is not always defined, but when it is, the usual axioms hold (see [22] for details). In this event, we can consider the B-completion $(G^A)^B$ of the partial B-group G^A which extends the given exponentiation by A (again see [22] for details). Since G is a torsion-free CSA group and A is a ring of characteristic zero the canonical homomorphism $G \to G^A$ is monic, and the completion G^A is again a torsion-free CSA group (see [22]). Now, it follows again from [22] that G^A canonically embeds into $(G^A)^B$. To finish the proof of the theorem we need to show that G^B is canonically B-isomorphic to $(G^A)^B$.

Let $\theta : G \longrightarrow H$ be a homomorphism of G into an arbitrary B-group H. The group H also can be viewed as an A-group, and, consequently, there exists a canonical A-homomorphism from G^A into H, This then can be extended to a B–homomorphism from the completion $(G^A)^B$ into the B–group H. So every homomorphism from G into H can be extended to a unique B–homomorphism from $(G^A)^B$ into H, i.e., $(G^A)^B$ is the B–completion of G, as desired.

Now, suppose that $A \leq B$ are torsion-free, unitary abelian groups. By Theorem A the rings $fr(A)$ and $fr(B)$ are unitary rings of characteristic zero. By Corollary 7 from **2.3** the inclusion $A \longrightarrow B$ extends to a monomorphism of unitary rings $fr(A) \longrightarrow fr(B)$. Finally, by Lemma 5, $G^A = G^{fr(A)}$ and $G^B = G^{fr(B)}$. The desired result is an immediate consequence of the discussion above.

Theorem B allows us to formulate the following result.

Proposition 3 *Let G be a non–abelian CSA-group and let A be a torsion–free, unitary abelian group. If H is a finitely generated subgroup of G^A then H is contained in a subgroup of the form G^{A_0}, where A_0 is a finitely generated unitary subgroup of A.*

Proof. The group $G^A = G^{fr(A)}$ is $fr(A)$-generated by G. If Y is a finite set of generators of $H \leq G^A$, then one needs just finitely many exponents from $fr(A)$ to be able to express all elements from Y in terms of A-generators from G. Denote this finite set of exponents by E. The group A generates the ring $fr(A)$; therefore there is a finitely generated subgroup $A_0 \leq A$ such that the subring $rg(A_0) \leq fr(A)$ contains the set E. By Corollary 2 of **2.1**, $rg(A_0)$ is isomorphic to $fr(A_0)$. It follows that H is contained in the $fr(A_0)$-subgroup of G^A, viewed as a $fr(A_0)$-group, generated by G; by Theorem B, this subgroup is $fr(A_0)$-isomorphic to $G^{fr(A_0)}$.

4 Separation, discrimination and CSA-groups.

We gather together in this section some simple results which will be of use in the sequel.

Recall that a group G is a CSA group if its maximal abelian subgroups are malnormal. As noted in the introduction, every abelian group is a CSA group; so too is every torsion-free hyperbolic group.

Lemma 7 *Let G be a torsion-free abelian group. If G separates a group H then G discriminates H.*

Proof. Suppose that the group G separates a group H. Then H can be embedded in a cartesian power P of G (see, for example [12]). Let $h_1, \ldots, h_n$ be non-trivial elements of H. Then there exists a projection π of P onto a finite direct product $G \times \ldots \times G$ such that $\pi(h_i) \neq 1$ for $i = 1, \ldots, n$. So, it suffices to prove that G discriminates a finite direct product $G \times \ldots \times G$. The result will follow if we prove that G discriminates $G \times G$. To this end let $(a_1, b_1), \ldots, (a_n, b_n)$ be non–trivial elements of $G \times G$. Since extraction of roots in a torsion–free abelian group is unique, whenever it is possible, there exists a positive integer

k such that the homomorphism from $G \times G$ into G defined by (note that we are using additive notation here)

$$(a, b) \mapsto a + kb \ (a, b \in G)$$

is monic on the set $\{(a_1, b_1), \ldots, (a_n, b_n)\}$. This completes the proof.

Lemma 8 *Let H be a non-abelian CSA group. If H is separated by a group G then H is discriminated by G.*

Proof. Let H be a non-abelian CSA group. We claim that for any non-trivial $a, b \in H$ there exists $x \in H$ such that $[a, b^x] \neq 1$. Indeed, suppose that a non-trivial element a commutes with all of the conjugates of a non-trivial b in H. Then it follows from the commutative transitivity of CSA groups (see [22]), that the normal closure B of b in H is abelian, i.e., H contains a non-trivial normal abelian subgroup B. Let M be a maximal abelian subgroup of H containing B (notice that $B \neq H$). Then M is not malnormal in H, which is impossible since H is a CSA-group.

Suppose now that H is separated by a group G and suppose that $h_1, \ldots, h_n$ are arbitrary non-trivial elements in H. There exists an element $x_1 \in H$ such that $[h_1, h_2^{x_1}] \neq 1$. Hence there exists an element $x_2 \in H$ such that $[[h_1, h_2^{x_1}], h_3^{x_2}] \neq 1$ and so on. It follows that we can find elements $x_1, x_2, \ldots, x_{n-1} \in H$ such that

$$c = [\ldots[[h_1, h_2^{x_1}], h_3^{x_2}], \ldots, h_n^{x_{n-1}}] \neq 1.$$

Therefore if ϕ is a homomorphism of H into G such that $\phi(c) \neq 1$. Then ϕ maps each of the elements $h_1, \ldots, h_n$ non-trivially into G, as desired.

5 Separation and extension of centralizers of hyperbolic groups

5.1 Direct extensions of centralizers and condition (S)

Let G be a group. In this subsection and the two subsequent ones, we will be concerned with various G-groups H obtained from G by means of *extending centralizers*. Our primary objective is to prove that many of these groups are often G–discriminated by G or some related supergroups of G.

Let G be a group. The centralizer of an element $u \in G$ is denoted by $C_G(u)$ or $C(u)$ or C, if the group G and the element u can be determined from the context.

We recall the definition introduced in the introduction in **1.2**.

Definition 4 *Let G be a group, $C(u)$ the centralizer of an element $u \in G$. Suppose that $C(u)$ is abelian and that $\phi : C(u) \longrightarrow A$ is a monomorphism of $C(u)$ into an abelian group A. Then the group*

$$G(u, A) = \langle G * A \mid C(u) = C(u)^{\phi} \rangle$$

is called an extension of the centralizer $C(u)$ by the group A with respect to ϕ.

We will, for the most part, regard such groups $G(u, A)$ as G-groups, with G here embedded in $G(u, A)$ by inclusion.

Some special instances of these $G(u, A)$ are of special interest. They depend on the nature of the embedding ϕ of $C(u)$ in A. We say that the extension is:

1) *direct* if $A = C(u)^{\phi} \times B$;

2) *free (of rank n)* if $A = C(u)^{\phi} \times B$ and B is a free abelian group (of rank n).

Free extensions of centralizers of rank 1 play an important role in this theory. Suppose then that $G = \langle X \mid R\rangle$ is a presentation of G. It is easy to see that a free rank 1 extension, which we denote by $G(u, t)$, of the centralizer $C(u)$ of the element u in the group G has the following presentation:

$$G(u, t) = \langle G,\ t \mid [y, t] = 1(y \in C(u))\rangle.$$

So here $A = C(u)\times < t >$. We will also sometimes use the following abbreviated description for $G(u, t)$:

$$G(u, t) = \langle G,\ t \mid [C(u), t] = 1\rangle.$$

Of course, $G(u, t)$ is an HNN-extension of G, with a single stable letter t, with associated subgroups $C(u)$ and $C(u)$ and associating isomorphism the identity.

Let C be a subgroup of the abelian group A and let $G(u, A)$ be an extension of the centralizer $C(u) = C$ of the group G by the group A. We say that an element $g \in G(u, A)$ is in *reduced form* if

$$g \ = \ g_1 a_1 \dots g_n a_n g_{n+1} \tag{2}$$

where $a_i \in A - C,\ i = 1, \dots, n$ and $g_j \in G - C,\ j = 2, \dots, n$.

In the case of a direct extension of the centralizer C by a group B, i.e., when $A = C \times B$, it is more convenient to use *semi-canonical forms*. Here a reduced form

$$g = g_1 b_1 \cdots b_n g_{n+1}$$

is termed *semi-canonical* if $1 \neq b_i \in B$ for all i. It is not difficult to see that one semi-canonical form of g can be transformed into another by a finite sequence of commuting relations which take the form $bc = cb$, where $c \in C,\ b \in B$.

In our treatment of extension of centralizers we will make use of some special G-homomorphisms of the kind described in the following lemma.

Lemma 9 *Let $G(u, A)$ be an extension of the centralizer $C = C(u)$ by an abelian group A. Let $\psi : A \longrightarrow C$ be a retraction of A onto its subgroup C. Then the homomorphism $\lambda_\psi : G(u, A) \longrightarrow G$, which is defined as the simultaneous extension to $G(u, A)$ of the identity homomorphism on G and the homomorphism ψ on A, is a G-homomorphism of the G-group $G(u, A)$ to the G-group G.*

The verification that λ_ψ has the stated properties is an immediate consequence of its definition.

The following lemma shows the importance of direct extensions of centralizers.

Lemma 10 *Let C be a subgroup of the torsion-free abelian group A. Then the following statements are equivalent:*

1. *A is C-discriminated by C;*

2. *$A = C \times B$ and B is discriminated by C.*

Proof. Suppose that 2) holds. Since B is discriminated by C, there is an associated family of discriminating homomorphisms Ψ from B to C. If we now denote by id_C the identity automorphism of C, then $\{id_C + m\psi \mid m \in \mathbf{Z}, \psi \in \Psi\}$ is a C-discriminating family of C-homomorphisms from $C \times B$ to C (see the proof of Lemma 7). This proves that 2) implies 1).

To prove that 1) implies 2) observe first that any C-homomorphism from A to C is a retract of A onto C. Since A is abelian, it follows that C is a direct summand of A and therefore we can express C in the form $C = A \times B$. Now A is C-discriminated by C. The associated family of C-homomorphisms of A onto C, restricted to B immediately show that B is discriminated by C, as required. This completes the proof of the lemma.

Our objective now is to obtain some conditions which ensure that certain extensions of centralizers $G(u, A)$ are G-discriminated by G. We will focus our attention first on the case where the extension is direct. With this in mind, let $u_1, \dots, u_n$ be elements of infinite order in a group G and let $w_1, \dots, w_{n+1}$ be arbitrary elements in G satisfying the following condition, which we term the (CS) condition:

$$[u_i^{w_{i+1}}, u_{i+1}] \neq 1, \quad i = 1, \dots, n-1.$$

We associate with such $U = (u_1, \dots, u_n)$ and $W = (w_1, \dots, w_n, w_{n+1})$ the sets

$$S(W, U, m) = \{w_1 {u_1}^{m_1} w_2 {u_2}^{m_2} \dots {u_n}^{m_n} w_{n+1} \mid |m_i| \geq m\}.$$

The following definition plays an important role in the rest of our development.

Definition 5 *We say that a group G satisfies the condition (S) if for every choice of elements $u_1, \dots, u_n$ and $w_1, \dots, w_{n+1}$ in G satisfying the condition (CS) there exists a constant m such that the set $S(W, U, m)$ does not contain the identity element.*

This approach goes back to [4], where it was proved that if G is a free group and if $u_1 = u_2 = \dots = u_n = u, u \neq 1$, then G satisfies the condition (S).

Notice that if G is a non-abelian CSA-group and if $u_1 = \dots = u_n = u$, the (CS) condition is equivalent to $[u, w_i] \neq 1$ for $i = 2, \dots n$. We observe also, for later use, that every torsion-free hyperbolic group satisfies the condition (S) ([24], [9]).

Now we return to the group $G(u, A)$, where $A = C \times B$. We observe that condition (S) in G allows us to approximate elements in B by large powers of elements in C, which in turn allows us to prove, under the right circumstances, that $G(u, A)$ satisfies a number of discrimination properties. In order to explain, we switch to additive notation, which makes sense because all of the groups involved are, for now, abelian. Thus, if B and C are abelian groups, then the set $Hom(B, C)$ can be turned into an abelian group under coordinatewise addition. So if $\mathcal{F}$ is a discriminating family of homomorphisms of B into C it makes sense to consider the subgroup $gp(\mathcal{F})$ of $Hom(B, C)$ generated by $\mathcal{F}$. We shall adopt this point of view and this notation throughout.

5.2 Discriminating direct extensions of centralizers

We are now in a position to formulate the main result about the discrimination of direct extensions of centralizers. As already stated, we will make use of the notation introduced above as well as that used in Lemma 9.

Theorem 1 *Let G be a CSA-group satisfying the condition (S), $G(u, A)$ a direct extension of the centralizer $C(u) = C$ of an element $u \in G$ of infinite order by an abelian group $A = C \times B$. If B is discriminated by C then $G(u, A)$ is G-discriminated by G. Moreover, $G(u, A)$ is again a CSA-group which satisfies the condition (S). Furthermore, if $\mathcal{F}$ is a C-discriminating family of homomorphisms of B into C, then $\{\lambda_\phi \mid \phi \in gp(\mathcal{F})\}$ is a G-discriminating family of homomorphisms of $G(u, A)$ into G.*

The proof of the theorem rests on the following two lemmas.

Lemma 11 *Let G be a group satisfying the condition (S). Then any group H which is discriminated by G also satisfies (S).*

Proof. Let tuples $W' = (w'_1, \ldots, w'_n, w'_{n+1})$ and $U' = (u'_1, \ldots, u'_n)$ of elements in H satisfy the condition (CS). We will prove that there exists an integer m such that all elements in the set $S(W, U, m)$ are non–trivial. By the (CS) condition, ${w'_{i+1}}^{-1} u'_i w'_{i+1}$ does not belong to the centralizer $C(u'_{i+1})$ of u'_{i+1} and hence the commutator $z_i = [{w'_{i+1}}^{-1} u'_i w'_{i+1}, u'_{i+1}]$ is non–trivial for each $i = 1, \ldots, n-1$. By hypothesis, H is G-discriminated by G. Hence there exists a G-homomorphism $\psi : H \longrightarrow G$ which maps all of the commutators z_i $(i = 1, \ldots, n-1)$ onto non-trivial elements of G. This implies that the images $\psi(W'), \psi(U')$ of W' and U', satisfy the condition (CS) in the group G. Observe that ψ maps $S(W', U', m)$ onto $S(\psi(W'), \psi(U'), m)$. Now G satisfies condition (S). Therefore there exists an integer m such that $S(\psi(W'), \psi(U'), m)$ does not contain the identity. Consequently, it's preimage $S(W', U', m)$ does not contain the identity. Thus the condition (S) holds in the group H.

The following lemma is the key step in the proof of Theorem 1.

Lemma 12 *Let G be a group satisfying the conditions of Theorem 1. Then $\{\lambda_\phi \mid \phi \in gp(\mathcal{F})\}$ G-discriminates the extension of the centralizer $G(u, A)$ by G.*

Proof. Let g be a non-trivial element in $G(u, A)$. One can write it in semi-canonical form

$$g = w_1 b_1 w_2 \cdots b_n w_{n+1},$$

where $1 \neq b_i \in B$, and $w_i \in G$ for all i and $w_i \notin C_G(u)$ for $i = 2, \ldots, n$.

Let $\psi \in \mathcal{F}$ separate the elements $b_1, \ldots, b_n$ in C. Put $u_i = \psi(b_i)$ for $i = 1, \ldots, n$. Let, as before, but using multiplicative notation now, $\lambda_{m\psi}$ be the homomorphism of $G(u, A)$ into G defined as the identity on G and the composition of ψ and the $m - th$ power map on B. Thus

$$\lambda_{m\psi}(b_i) = u_i^m \in C.$$

Since the centralizer of a non-trivial element in a non-abelian CSA-group is malnormal, the sets $W = (w_1, \ldots, w_n + 1)$ and $U = (u_1, \ldots, u_n)$ satisfy the separation condition (CS). So, by the (S)-condition, the set $S(W, \lambda_{m\psi}(b_1, \ldots, b_n)$ does not contain any trivial elements for some large enough m. Therefore $\lambda_{m\psi}$ maps g non–trivially into G. Thus, we can separate any given non–trivial element $g \in G(u, A)$ by $\lambda_{m\psi}$ if m is large enough, say $m \geq m(g)$. Consequently, if we have a finite number of elements $g_1, \ldots, g_k \in G(u, A)$, then $\lambda_{m\psi}$ will separate them all if $m \geq max\{m(g_1), \ldots, m(g_k)\}$. This means that $G(u, A)$ is G–discriminated by G.

It follows now that in order to complete the proof of Theorem 1, we need only note that $G(u, A)$ is a CSA-group, which was proved in [22].

5.3 General extensions of centralizers

Extensions of centralizers are not always direct. We need some terminology to deal with arbitrary extensions.

Recall that if C is a subgroup of an abelian group A then we denote by $\sqrt{C}$ the isolator of C in A, i.e., the preimage in A of the torsion subgroup of A/C. If $\sqrt{C} = C$, then C is isolated in A.

We will adopt this notation throughout, as well as the notation that we introduced earlier. In keeping with the definitions recorded in the introduction, we make also the following

Definition 6 *$G(u, A)$ is said to be of type L if $C = C(u)$ is an isolated subgroup of A; $G(u, A)$ is said to be of type B if $\sqrt{C} = A$.*

Notice that $G(u, \sqrt{C})$ is an extension of the centralizer C of type B and, assuming that $C_{G(u,\sqrt{C})}(u) = \sqrt{C}$ in $G(u, \sqrt{C})$, $G(u, A) = G(u, \sqrt{C})(u, A)$ can be viewed as an extension of the centralizer $C_{G(u,\sqrt{C})}(u)$ of type L. Under these circumstances, we introduce the following definition.

Definition 7 *The sequence of extensions*

$$G \leq G(u, \sqrt{C}) \leq G(u, A)$$

is called the canonical B-L decomposition.

The next lemma details a number of simple situations where $\sqrt{C}$ is a direct factor of A.

Lemma 13 *Let C be a subgroup of the torsion-free abelian group A. Then the following hold:*

1. *If A/C is a free abelian group then $A = C \times B$ and $C = \sqrt{C}$;*

2. *If A is finitely generated, then $A = \sqrt{C} \times B$ and B is free of finite rank;*

3. *If C is divisible, then $A = C \times B$ and $C = \sqrt{C}$;*

4. *If $\sqrt{C}$ is finitely generated and A is free, then $A = \sqrt{C} \times B$.*

The proof of Lemma 13 is straightforward and is left to the reader.

We have then the following corollary of Theorem 1 and Lemma 13 concerning extensions of centralizers of type L.

Corollary 10 *Let G be a CSA-group satisfying condition (S), let A be a torsion-free abelian group and let $G(u, A)$ be an extension of type L of the centralizer $C_G(u)$ of an element $u \in G$ of infinite order in G. Then the following hold:*

1. *(Global approximation) If A is free abelian and $C(u)$ is finitely generated, then $G(u, A)$ is G-discriminated by G;*

2. *(Local approximation) Every finitely G-generated G-subgroup of $G(u, A)$ is G-discriminated by G.*

Proof. In order to prove 1), notice that, by Lemma 13, the given extension is free. The conclusion follows immediately from Theorem 1.

In order to prove 2), observe that any finitely G-generated G-subgroup H of $G(u, A)$ is contained in a subgroup $G(u, B)$, where B is a finitely generated subgroup of A. In this case B is a free abelian group and, by 1) above, $G(u, B)$ is G-discriminated by G. Hence so too is its G-subgroup H.

Under certain circumstances, the condition (S) persists under extensions of centralizers, as the following theorem shows.

Theorem 2 *Let G be a torsion-free hyperbolic group and let A be a torsion-free abelian group. Then $G(u, A)$ satisfies the condition (S).*

Proof. Notice that in a torsion-free hyperbolic group, centralizers of non-trivial elements are infinite cyclic. Therefore $C = C(u)$ is generated by v, say, and $G(u, A)$ is the amalgamated product of G and A with the element v identified with an element $a \in A$. Now the condition (S) is a local condition, i.e., it is satisfied if and only if it is satisfied in every finitely generated subgroup. It suffices therefore to prove that $G(u, B)$ satisfies the condition (S), where B is a finitely generated subgroup of A containing C, i.e., the element a. Let $D = \sqrt{C}$ be the isolator of C in A. Then D is again cyclic, generated say by b and $a = b^n$ for some integer $n \neq 0$. Thus $G(u, \sqrt{C})$ is simply the amalgamated product of the hyperbolic group G and the infinite cyclic group generated by b with v identified with b^n. Consequently, by a theorem of [15], $G(u, \sqrt{C})$ is hyperbolic and so, as we mentioned in **5.1**, satisfies (S). The canonical B-L-decomposition allows us to view $G(u, B)$ as $G(u, \sqrt{C})(u, B)$ which is of type L, indeed a free extension of centralizers. Consequently, by Corollary 10, (1), the $G(u, \sqrt{C})$-group $G(u, B)$ is $G(u, \sqrt{C})$-discriminated by $G(u, \sqrt{C})$. Since, as noted, $G(u, \sqrt{C})$ satisfies (S), so too does $G(u, B)$, by Lemma 11. This completes the proof.

We now put Theorem 1, Theorem 2 and Lemma 7 together in order to prove the following

Theorem 3 *(Global approximations for arbitrary extensions) Let G be a torsion-free hyperbolic group, let A be a torsion-free abelian group and let $B = A \otimes_{\mathbb{Z}} \mathbb{Q}$. Then for every non-trivial element u of G the extension of centralizer $G(u, A)$ is G-discriminated by the G-group $G(u, \mathbb{Q})$.*

Proof. We note that since A is torsion-free abelian, the mapping $a \mapsto a \otimes 1$ ($a \in A$) is an embedding of A in B. The embedding of the cyclic centralizer $C_G(u)$ in A can be extended to an embedding of the additive group $\mathbb{Q}$ of the rationals into B which in turn gives rise to an embedding of $G(u, \mathbb{Q})$ into $G(u, B)$. It suffices then to prove that the G-group $G(u, B)$ is $G(u, \mathbb{Q})$-discriminated by $G(u, \mathbb{Q})$.

Now $\mathbb{Q}$ is a divisible subgroup of the abelian group B. Hence, it is a direct factor of B. Moreover, the divisible torsion-free abelian group B is isomorphic to a direct sum of copies of $\mathbb{Q}$; hence B is separated by $\mathbb{Q}$ and by Lemma 7 B is discriminated by $\mathbb{Q}$. It follows then from Theorem 2 and Theorem 1 that $G(u, B)$ is $G(u, \mathbb{Q})$-discriminated by $G(u, \mathbb{Q})$, as required.

It follows from the proof of Theorem 3 that the following corollary also holds.

Corollary 11 *(Global approximations for divisible extensions) Let G be a torsion-free hyperbolic group and let A be a torsion-free, divisible abelian group. Then $G(u, \mathbb{Q})$ can be viewed as a subgroup of $G(u, A)$ and $G(u, A)$ is $G(u, \mathbb{Q})$-discriminated by $G(u, \mathbb{Q})$.*

Finally we observe the further corollary.

Corollary 12 *(Local approximations) Let G be a torsion-free hyperbolic group and let $G(u, A)$ be an arbitrary A-extension of a centralizer $C(u)$ by a torsion-free abelian group A. Then every finitely generated $G(u, \sqrt{C})$-subgroup of $G(u, A)$ is $G(u, \sqrt{C})$-discriminated by $G(u, \sqrt{C})$.*

Proof. A finitely generated $G(u, \sqrt{C})$-subgroup H of $G(u, A)$ is contained in some subgroup $G(u, A_0)$, where A_0 is a finitely generated subgroup of A containing $\sqrt{C}$. Now $G(u, \sqrt{C})$ is simply the result of adjoining to G an n-th root of the generator of the infinite cyclic group C and so it is hyperbolic [15]. It follows that $G(u, A_0)$ is an extension of type L of the hyperbolic group $G(u, \sqrt{C})$. Hence by the local approximation theorem for type L extensions, we find that $G(u, A_0)$ is $G(u, \sqrt{C})$-discriminated by $G(u, \sqrt{C})$.

6 Discriminating exponential groups

The objective of this section is to prove our main result, which takes the following form:

Theorem 4 *Let G be a torsion-free hyperbolic group and let $A \leq B$ be unitary rings of characteristic zero. Suppose that $gp(1)$ is a direct summand of A^+ and that the additive group B^+ is A^+-discriminated by its subgroup A^+. Then G^B is G^A-discriminated by G^A.*

We begin the preparations for the proof of Theorem 4 with the following

Proposition 4 *Let G be a torsion-free hyperbolic group and let A be an unitary ring of characteristic zero. Then G^A is a CSA group which satisfies the separation condition (S).*

Proof. It was proved in [22] that any completion of a torsion-free CSA group by a ring of characteristic zero is CSA. Hence G^A is a CSA group. Since G^A is a union of a sequence of extension of centralizers, G^A satisfies the separation condition (S), by Theorem 2, as required.

The proof of Theorem 4 depends not only on Proposition 4, but on two additional lemmas. We need to introduce some additional notation. To this end, suppose that B is a unitary ring and that x is an element not in any of the subgroups or rings under consideration. Then we define

$$x^B = \{x^\beta \mid \beta \in B\},$$

which we turn into a B-group by defining

$$x^1 = x, \quad x^\beta x^{\beta'} = x^{\beta+\beta'}, \quad (x^\beta)^{\beta'} = x^{\beta\beta'}, \quad \beta, \beta' \in B.$$

x^B is simply a multiplicative copy of B^+ and so is a unitary abelian group with x its distinguished element. If A is a unitary subring of the unitary ring B, then we can think of x^A is a unitary subgroup of the unitary abelian group x^B. Notice that the B-subgroup of the B-group x^B generated by x is x^B. More generally, if H is a B-group, then the B-subgroup G of H generated by an element $h \in H$ consists of all elements of the form h^β, where β ranges over B; we denote this subgroup by h^B.

We are now in a position to prove the following lemma (see the proof of Lemma 11).

Lemma 14 *Let A be a unitary ring of characteristic 0 and let H be a torsion-free CSA A-group satisfying the condition (S). Suppose that H is a subgroup of a group J and that J is H-discriminated by H. Furthermore, suppose that A is the direct sum of $\mathbb{Z} = gp(1)$ and A_1, and the centralizer of $a \in J$ is generated by a. Then for any x the generalized free product*

$$I = < J * x^A; a = x >$$

is H-discriminated by H.

A few words of explanation might be helpful here. The main point is that I is obtained from J by extending the centralizer of a by x^A. Notice that x^A is the direct product of an infinite cyclic group and x^{A_1}:

$$x^A = x^{\mathbb{Z}} \times x^{A_1}.$$

Notice also that the infinite cyclic subgroup $gp(a)$ is identified with the subgroup $x^{\mathbb{Z}}$ by identifying a with x.

Proof. Let $\mathcal{F}$ be a discriminating family of H-homomorphisms from J onto H. For each positive integer n and each homomorphism $\phi \in \mathcal{F}$, let ϕ_n be the H-homomorphism from I to H defined as follows:

$$\phi_n(j) = \phi(j),\ \phi_n(x^{m+\alpha_1}) = \phi(a)^{m+n\alpha_1}\ (j \in J,\ m \in \mathbb{Z},\ \alpha_1 \in A_1).$$

Notice that $\phi_n(a) = \phi(a)$ and $\phi_n(x) = \phi(a)$ and therefore the definition of ϕ_n makes sense. Notice also that if $\alpha_1 \in A_1$ and $\alpha = m1 + \alpha_1 \in A$ then

$$\phi_n(x^\alpha) = \phi(a^m)(\phi(a)^{\alpha_1})^n.$$

Observe, in particular, that if $b \in x^{A_1}$, then $\phi_n(b)$ is an n-th power of a power of $\phi(a)$. We claim that the family

$$\mathcal{F}_n = \{\phi_n \mid \phi \in \mathcal{F},\ n = 1, 2, \ldots\}$$

is a discriminating family of H-homomorphisms of I onto H. With this in mind, let $c \in I, c \neq 1$. If $c \in J \cup x^A$, there exists an element in $\mathcal{F}_n$ which maps c into a non-trivial element of H. In all other cases, c can be written in the form

$$c = a_1 b_1 a_2 \ldots a_k b_k a_{k+1},$$

where

$$a_i \in J, b_i \in x^{A_1},\ [a_i, a] \neq 1\ (2 \leq i \leq k),\ b_i \neq 1.$$

Since $\mathcal{F}$ is an H-discriminating family of H-homomorphisms of J onto H, we can find an element $\phi \in \mathcal{F}$ such that

$$[\phi(a), \phi(a_i)] \neq 1\ (2 \leq i \leq k).$$

Expressing each b_i in the form $b_i = x^{\alpha_i}$, where α_i is a non-zero element of A_1, it follows from the transitivity of commutation in H, that

$$[\phi(a)^{\alpha_i}, \phi(a_i)] \neq 1\ (2 \leq i \leq k).$$

Since H is a CSA-group, it follows (see the proof of Lemma 11) that

$$[\phi(a_{i+1})^{-1}\phi(a)^{\alpha_i}\phi(a_{i+1}), \phi(a)^{\alpha_{i+1}}] \neq 1\ (1 \leq i \leq k-1).$$

Hence the tuples $\phi(a_1), \ldots, \phi(a_{k+1})$ and $\phi(a)^{\alpha_1}, \ldots, \phi(a)^{\alpha_k}$ satisfy the condition (CS). Since H satisfies the separation condition (S),

$$\phi_n(c) = \phi(a_1)(\phi(a)^{\alpha_1})^n \phi(a_2) \ldots \phi(a_k)(\phi(a)^{\alpha_k})^n \phi(a_{k+1}) \neq 1$$

for all sufficiently large values of n. It follows then, in the same way, that for any finite set of non-trivial elements of I, there exists an element $\psi \in \mathcal{F}$ and an integer n such that ψ_n separates these elements in H, as required.

It is worth emphasizing here that every $\phi \in \mathcal{F}$ extends to an H-homomorphism ϕ^+ of I onto H; we say that ϕ^+ continues ϕ.

Finally we prove one more lemma before embarking on the proof of Theorem 4.

Lemma 15 *Let δ be a limit ordinal and let*

$$G = G_0 \leq G_1 \leq \ldots \leq G_\alpha \leq G_{\alpha+1} \leq \ldots \leq G_\delta$$

be a chain of groups such that for every limit ordinal $\alpha \leq \delta$,

$$\bigcup_{\beta<\alpha} G_\beta = G_\alpha.$$

Suppose that for every ordinal $\alpha < \delta$, every G-homomorphism from G_α onto G can be continued to a G-homomorphism from $G_{\alpha+1}$ onto G. Then the following hold:

1. *for every $\alpha < \delta$, every G-homomorphism $\phi : G_\alpha \longrightarrow G$ can be continued to a G-homomorphism from G_δ onto G;*

2. *if G_α is G-separated by G for every $\alpha < \delta$ then G_δ is G-separated by G;*

3. *if G_α is G-discriminated by G for every $\alpha < \delta$ then G_δ is G-discriminated by G.*

Proof. 1. Let $\phi : G_\alpha \longrightarrow G$ be a G-homomorphism from G_α onto G. Put $\phi_\alpha = \phi$. Suppose now that $\alpha < \beta < \delta$ and that we have defined for every choice of $\alpha < \gamma < \beta$, a family of G-homomorphisms $\phi_\gamma : G_\gamma \longrightarrow G$ such that if $\alpha < \gamma' < \gamma < \beta$, then ϕ_γ agrees with $\phi_{\gamma'}$ on $G_{\gamma'}$. If β is a limit ordinal, then de define ϕ_β to be the "union" of the ϕ_γ for $\gamma < \beta$, i.e., if $u \in G_\beta$, then $u \in G_\gamma$ for some $\gamma < \beta$. Define $\phi_\beta(u) = \phi_\gamma(u)$, which makes sense because of our assumptions. If β is not a limit ordinal, it can be written in the form $\beta = \beta' + 1$. Define ϕ_β to be any extension of $\phi_{\beta'}$ to G_β. We then define ϕ_δ to be the union of the ϕ_β with $\beta < \delta$. This completes the proof.

The proofs of 2 and 3 follow immediately from the proof of 1.

We are now in a position to prove Theorem 4. We can assume that $G \neq 1$. Put $H = G^A$. By Proposition 4, $G^A = H$ is a CSA-group satisfying the condition (S). It follows from [21] that G^B can be obtained as the union of a chain of groups

$$H = H_0 < H_1 < \ldots < H_\alpha < \ldots < H_\delta = G^B$$

satisfying the following conditions:

1. δ is a limit ordinal;

2. if $\alpha \leq \delta$ is a limit ordinal, then

$$H_\alpha = \bigcup_{\alpha'<\alpha} G_{\alpha'};$$

3. if $\alpha = \alpha'+1$, then H_α is obtained from $H_{\alpha'}$ by an extension of a centralizer C of an element $h \in H_{\alpha'}$ and C takes on one of the following two forms:

3(i). $C = h^A$;

3(ii). $C = h^{\mathbb{Z}}$.

Now suppose that for each $\alpha' < \alpha$ we have proved

(a) $H_{\alpha'}$ is H-discriminated by H

(b) if $\alpha'' < \alpha' < \alpha$, then every H-homomorphism from $H_{\alpha''}$ onto H can be extended to an H-homomorphism from $H_{\alpha'}$ onto H.

If α is a limit ordinal, it follows from Lemma 15 that H_α is H-discriminated by H.

If $\alpha = \alpha' + 1$, then there are the two cases 3(i) and 3(ii) to consider. In case 3(i), H_α is $H_{\alpha'}$-discriminated by $H_{\alpha'}$ and since $H_{\alpha'}$ is H-discriminated by H, H_α is H-discriminated by H. Moreover, it follows from the proof of Theorem 1 (or directly) that every H-homomorphism from $H_{\alpha'}$ to H can be continued to an H-homomorphism from H_α to H.

In case 3(ii) it follows from Lemma 14, that H_α is H-discriminated by H and again every H-homomorphism from $H_{\alpha'}$ to H can be continued to an H-homomorphism from H_α to H. This completes the proof of Theorem 4 on applying Lemma 15.

The theorems discussed in the introduction are all consequences of Theorem 4.

Theorem C1 *Let G be a torsion-free hyperbolic group and let $\mathbb{Z}[x]$ be the ring of integral polynomials in a single variable x. Then $G^{\mathbb{Z}[x]}$ is G- discriminated by G.*

Proof. Put $A = \mathbb{Z}$ and $B = \mathbb{Z}[x]$. Then B is discriminated by A and $gp(1) = A$ (in particular, $gp(1)$ is a direct summand of A). Now the result follows from Theorem 4.

Theorem C2 *Let G be a torsion-free hyperbolic group and let A be a unitary ring with free abelian additive group A^+. Then any A-completion G^A of type L of the group G (as $\mathbb{Z}$-group) is G-discriminated by G.*

Proof. Notice, that since A^+ is a free abelian group and $A^+/gp(1)$ is torsion-free, $gp(1)$ is a direct summand of A^+. Hence the rings $\mathbb{Z} \leq A$ satisfy the conditions of Theorem 4. Consequently, G^A is G-discriminated by G.

One more application of Theorem 4 comes from completions of hyperbolic groups by unitary abelian groups. Suppose $A \leq B$ are torsion-free , unitary abelian groups, that $gp(1)$ is a direct summand of A^+ that B is A-discriminated by A. Now $G^A = G^{fr(A)}$ and $G^B = G^{fr(B)}$. It follows from the conditions laid down in **2** that $fr(B)^+$ is $fr(A)^+$- discriminated by $fr(A)^+$. Hence Theorem 4 applies and so we have proved the following theorem.

Theorem C3 *Let G be a torsion-free hyperbolic group and let A be a torsion-free unitary abelian group. If A is free abelian and if $gp(1)$ is isolated in A, then G^A is G-discriminated by G.*

References

[1] Bass H., *Groups acting on non–Archimedian trees.* Arboreal group theory, (1991) pp. 69–130.

[2] Baumslag B., *Residually free groups.* Proc.London Math. Soc., (1967) 17 (3), pp.402–418.

[3] Baumslag G., *Some aspects of groups with unique roots.* Acta Math., (1960) 104, pp. 217–303.

[4] Baumslag G., *On free* **Q***-group.* Comm. Pure and Appl. Mathematics, (1965) 18, pp.25–30.

[5] Baumslag G., *On generalized free products.* Math. Zeitschr., (1962) 7, 8, pp. 423–438.

[6] Baumslag G., Myasnikov A., Remeslennikov V., *Residually hyperbolic groups.* Proc. Inst. Appl. Math. Russian Acad. Sci., 1995, 24, p.3-37.

[7] Baumslag G., Myasnikov A., Remeslennikov V., *Algebraic geometry over groups I: Algebraic sets and ideal theory.* To appear in J.of Algebra.

[8] Baumslag G., Myasnikov A., Remeslennikov V., *Discriminating and co-discriminating groups.* To appear in Journal of Group Theory.

[9] Baumslag G. and Short H., *A remark on hyperbolic groups.* Unpublished.

[10] Bestvina M., and Feighn M., *A combination theorem for negatively curved groups.* J. Diff. Geom., (1992)35, pp.85–101.

[11] Hall P., *Nilpotent groups.* Queen Mary College Math. Notes (1979).

[12] Hall P., *Finiteness conditions for infinite soluble groups.* Proc. London Math. Soc. (1954).

[13] Gildenhuys D., Kharlampovich O., and Myasnikov A., *CSA groups and separated free constructions.* Bull. Austr. Math. Soc., 1995, v. 52, 1, p.63-84.

[14] Kaplansky I., *Infinite abelian groups.* Ann Arbor: Univ. of Michigan Press, 1954.

[15] Kharlampovich O. and Myasnikov A., *Hyperbolic groups and free constructions.* Transactions of Math., 350(1998), 2, pp.571-613.

[16] Kurosh A.G., *The theory of groups*, I, II, 2nd Edition (translated and edited by K.A. Hirsch), Chelsea, New York (1960).

[17] Lioutikov S. and Myasnikov A., *Centroids of groups.* Journal of Group Theory, v.3, iss.2 (2000), pp. 177-197.

[18] Lyndon R.C., *Groups with parametric exponents.* Trans. Amer. Math. Soc., (1960) 9, 6, pp.518–533.

[19] Lyndon R.C., and Schupp P.E., *Combinatorial group theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete 89, Springer-Verlag, Berlin, Heidelberg, New York (1977).

[20] Makanin G.S., *Decidability of the universal and positive theories of a free group* . Math. USSR Izvestiya, (1985) 25(1), pp.75-87.

[21] Myasnikov A., Remeslennikov V., *Exponential groups I: foundations of the theory and tensor completion.* Siberian Math. J., (1994) 5.

[22] Myasnikov A., Remeslennikov V., *Exponential groups II: extensions of centralizers and tensor completion of CSA-groups.* International Journal of Algebra and Computation, 6(6):687–711, 1996.

[23] Neumann B.H., *An essay on free products of groups with amalgamations.* Philos. Trans. Royal Society London, Ser. A 246 (1954), pp. 503–554.

[24] Ol'shanskii A. Yu. *On residualing homomorphisms and G-subgroups of hyperbolic groups.* Int. J. Algebra and Computation, 1993, 3(4), pp.365–409.