Discriminating and co-discriminating groups

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1 Introduction

1.1 Separation conditions

This paper is concerned with proving that a number of groups satisfy certain separation conditions. In order to explain what these conditions are we need to recall some definitions from [5] (see also [4]).

Definition 1 *Let H be a group, $\mathcal{S}$ a non-empty set of groups and $\mathcal{F}$ a family of homomorphisms of H into the groups in $\mathcal{S}$. We term $\mathcal{F}$ a separating family if for each $h \in H$, $h \neq 1$, there exists $\phi \in \mathcal{F}$ such that $\phi(h) \neq 1$. In this event we also say that $\mathcal{S}$ separates H or that H is residually $\mathcal{S}$.*

If $\mathcal{S}$ consists of the singleton G, then we say that G *separates* H or that H is *residually* G.

Definition 2 *Let H be a group, $\mathcal{S}$ a non-empty set of groups and $\mathcal{F}$ a family of homomorphisms of H into the groups in $\mathcal{S}$. We term $\mathcal{F}$ a discriminating family if for each finite subset Y of non-trivial elements of H, there exists a homomorphism $\phi \in \mathcal{F}$ such that $\phi(h) \neq 1$ for every $h \in Y$ (we say that ϕ discriminates the finite set Y). In this event we say that $\mathcal{S}$ discriminates H.*

If $\mathcal{S}$ consists of a singleton G we say that G *discriminates* H, or that H is *fully residually* G or that H is ω*–residually* G.

These notions of separation and discrimination play a role in several areas of group theory; for example, in the theory of varieties of groups [17], in algorithmic group theory [14] and in, what we have termed, algebraic geometry over groups [5]. Our concern here is with two related notions. However, before we turn our attention to them, it may be worth noting that there is a distinct difference between the notion of separation and that of discrimination. The interested reader will find it instructive to reflect on the notion of a root property discussed in the paper [7], the results in [2], [3], [6] and the book [17].

Our concern here will be with two notions related to separation and discrimination.

1.2 Discriminating groups

The first of these is that of a *discriminating* group. We term a group G *discriminating* if every group separated by G is discriminated by G. It is not easy to determine which groups are discriminating, although it is is not hard to find a characterization of such groups. Indeed *a group G is discriminating if and only if its direct square $G \times G$ is discriminated by G* (see section **2**). We concentrate here on the task of classifying those abelian groups which are discriminating. It turns out that every torsion–free abelian group is discriminating. Our main result here, Theorem A, involves torsion abelian groups. We refer the reader to the books (see [9], [11]) for a comprehensive discussion of abelian groups and the notions described below. In order to formulate Theorem A we need to digress a little. Let then A be a torsion abelian group. Given an integer n, we define

$$A[n] = \{a \in A \mid a^n = 1\}.$$

Clearly, $A[n]$ is a subgroup of A. Observe that $A[p^{k-1}] \leq A[p^k]$, for every prime p and every positive integer k. Moreover $A[p^k]/A[p^{k-1}]$ is of exponent p and can therefore be viewed as a vector space over the field of p elements. We denote the dimension of a vector space V by $dim(V)$ and define

$$\alpha(A, p, k) = dim(A[p^k]/A[p^{k-1}]).$$

Consider next the Sylow p-subgroup $\tau_p(A)$ of A. Its maximal divisible subgroup $\delta(\tau_p(A))$ is a direct sum of a number of quasi-cyclic groups of type Z_{p^∞}. The cardinality of the set of summands involved is an invariant of A, which we denote by $\beta(A, p)$. These invariants can be traced back to Ulm (see [9]); they were explicitly introduced in [18]. Finally, we recall that $a \in A$ *has infinite* $p-height$ if the equation $x^{p^k} = a$ has a solution in A for every positive integer k. We are now in a position to formulate

Theorem A *Let A be a torsion, abelian group and suppose that $\tau_p(A)/\delta(\tau_p(A))$ has no elements of infinite p-height. Then A is discriminating if and only if $\alpha(A, p, k)$ is either zero or infinite for every prime p and every positive integer k and $\beta(A, p)$ is either zero or infinite for every prime p.*

1.3 Co-discriminating groups

Dually, we term a group H *co-discriminating* if an arbitrary family $\mathcal{S}$ of groups separates H if and only if $\mathcal{S}$ discriminates H. We will focus here on a large class of co-discriminating groups, *groups without zero divisors*, which were introduced in [5] in a rather different context. We recall that a group H has no zero divisors if given any non-trivial elements $a, b \in H$ there exists an element $x \in H$ such that $[a, b^x] \neq 1$. Groups without zero divisors are called *domains* and, as noted above (and easily proved directly from the definition) every such domain is co-discriminating. These domains include, in particular, the *CSA groups* introduced in [5]. We recall that a group G is a *CSA group* if every maximal abelian subgroup M of G is malnormal, i.e., $M^g \cap M = 1$ for every $g \in G - M$. The class of CSA-groups is quite substantial. It includes all abelian groups, all torsion-free hyperbolic groups [16] and all groups acting freely on Λ-trees [1] as well as many one-relator groups – a complete description of one-relator CSA-groups was obtained by D. Gildenhuys, O. Kharlampovich and A. Myasnikov in [10]. It turns out that many free products with amalgamation and many one-relator groups are domains. In particular, a free product $A * B$ is a domain unless both the groups A and B are of order 2. This allows one to view a result of B. Baumslag [2] that a free product $A * B$ is separated by a non-abelian free group if and only if both the groups A and B are discriminated by a non-abelian free group in a slightly different light. This theorem of Baumslag follows immediately from the remarks above. Indeed, more generally, the corresponding result holds when the non-abelian free group is replaced by an arbitrary torsion-free group G which separates its own free square $G * G$, for instance every non-abelian torsion-free hyperbolic group (see [4]).

The main result of the paper is

Theorem B *Every one-relator group with more then 2 generators is a domain, in particular it is co-discriminating.*

It follows, perhaps surprisingly (see Section 4.3, Theorem 6), that every one–relator group which is separated by a free group is discriminated by a free group.

2 Discriminating abelian groups

2.1 Discriminating groups

Recall that a group G is *discriminating* if every group separated by G is discriminated by G.

First we record the following well-known lemma, the proof of which is left to the reader (see [8]).

Lemma 1 *A group H is separated by a family of groups S =$\{G_i \mid i \in I\}$ if and only if H is a subgroup of a cartesian product of groups from $\mathcal{S}$.*

The following criterion is useful.

Proposition 1 *A group G is discriminating if and only if $G \times G$ is discriminated by G.*

Proof. Suppose that the group $G \times G$ is discriminated by G and that a group H is separated by G. Now by Lemma 1, H can be embedded in a cartesian power P of G. Let $h_1, \ldots, h_n$ be a finite number of non-trivial elements of H. Then there exists a projection π of P onto a finite direct product $G^k = G \times \ldots \times G$ of k copies of G such that $\pi(h_i) \neq 1$ for $i = 1, \ldots, n$. Since G discriminates $G \times G$ it follows readily that G discriminates G^k. Thus 2 implies 1. The reverse implication is immediate.

We make use of this criterion repeatedly in what follows.

We gather together a number of results about discriminating groups. Throughout this section, p will denote a prime.

Proposition 2 *Let D be a discriminating group. Then the following conditions hold.*

1. *If H is a finite subgroup of $D \times D$, then there is a finite subgroup of D which is isomorphic to H.*

2. *If H is a finite subgroup of $D \times D$ and H is contained in a divisible subgroup (i.e., extraction of n-th roots is possible in that subgroup for every positive integer n) of $D \times D$, then there is a finite subgroup of D which is contained in a divisible subgroup and is isomorphic to H.*

3. *If H is a finite subgroup of a Sylow p–subgroup (i.e., a subgroup in which every element has order a power of p and which is maximal subject to this condition) of $D \times D$, then there is a finite subgroup of a Sylow p–subgroup of D which contains a subgroup isomorphic to H.*

4. *The number of elements of a given finite order in D is either zero or infinite.*

5. *If D_i is discriminating for every $i \in I$, then the cartesian product $\overline{\prod}_{i\in I} D_i$ and the restricted direct product $\prod_{i\in I} D_i$, are discriminating.*

Proof. If D is discriminating, then for every finite subgroup K of $D \times D$ there exists a homomorphism $\phi : D \times D \longrightarrow D$ which is monic on K. If a finite subgroup K of $D \times D$ is contained in a divisible subgroup or a Sylow p-subgroup of $D \times D$ then it's image under ϕ has the same property. These remarks are enough for the proof of 1, 2, 3 and 4.

We prove next that $D = \overline{\prod}_{i\in I} D_i$ is discriminating – the proof that $\prod_{i\in I} D_i$ is discriminating is similar and is left to the reader. For each $i \in I$ we choose a discriminating family of homomorphisms Φ_i from $D_i \times D_i$ into D_i. For any "I-sequence" of homomorphisms

$$\phi \in \overline{\prod}_{i\in I} \Phi_i$$

we define a homomorphism $\lambda_\phi : D \times D \longrightarrow D$ as follows: if $f, h \in D$ then

$$\lambda_\phi(f, h)(i) = \phi_i(f(i), h(i)), \quad i \in I.$$

Clearly, λ_ϕ is a homomorphism of $D \times D$ into D. Now, if we have finitely many non–trivial elements $(f_1, h_1), \ldots, (f_n, h_n) \in D \times D$, then we can find a finite subset I_0 of I such that for each of the elements (f_k, h_k) there exists an index $i \in I_0$ such that $(f_k(i), h_k(i)) \neq 1$ in $D_i \times D_i$. Since D_i is discriminating there exists a homomorphism $\phi_i \in \Phi_i$ discriminating all the elements $(f_k(i), h_k(i))$ for $i \in I_0$. If, for each $i \notin I_0$, we choose ϕ_i to be any element in Φ_i, we obtain an I-sequence $\phi \in \overline{\prod}_{i\in I} \Phi_i$. Obviously, the homomorphism λ_ϕ discriminates all of the elements (f_k, h_k), which proves that D discriminates $D \times D$.

2.2 Abelian groups

Next we focus on the problem as to which abelian groups are discriminating. We shall use additive notation throughout this subsection.

If A is an abelian group, then we denote the torsion subgroup of A by $\tau(A)$ and, for each prime p, we denote its unique Sylow p–subgroup by $\tau_p(A)$. Notice that $\tau(A)$ is the direct sum of its Sylow p–subgroups:

$$\tau(A) = \oplus_p \tau_p(D).$$

Finally, let $\delta(A)$ be the maximal divisible subgroup of A. A is termed *reduced* if $\delta(A) = 0$. Every divisible subgroup of an abelian group is a direct summand; so $\delta(A)$ is a direct summand of A and therefore we can express A in the form $A = B \oplus \delta(A)$. It follows that $B \cong A/\delta(A)$ is reduced. Notice also, that every divisible abelian p–group is a direct sum of groups of type $Z(p^\infty)$, i.e., additive copies of the multiplicative group of all p^n–th roots of unity. The following proposition then holds.

Proposition 3 *Let A be an abelian group.*

1. *If A is torsion-free abelian, then A is discriminating.*
2. *If A is discriminating, then so too is its torsion subgroup $\tau(A)$.*
3. *If A is a torsion group, then A is discriminating if and only if all of its Sylow p–subgroups $\tau_p(A)$ are discriminating.*
4. *If A splits over $\tau(A)$, then A is discriminating if and only if $\tau(A)$ is discriminating.*
5. *If A is a non–trivial divisible p–group, then A is discriminating if and only if it is a direct sum of infinitely many copies of $Z(p^\infty)$.*
6. *If A is discriminating then so too is $\delta(A)$.*

Proof.

1. Let $(a_1, b_1), \ldots, (a_n, b_n)$ be non–trivial elements of $A \oplus A$. Since extraction of roots in a torsion–free abelian group is unique, whenever it is possible, there exists a positive integer k such that the homomorphism from $A \oplus A$ into A defined by

$$(a, b) \mapsto a + kb \ (a, b \in A)$$

is monic on the set $\{(a_1, b_1), \ldots, (a_n, b_n)\}$. This completes the proof.

2. It follows immediately from the fact that $A \oplus A$ is discriminated by A, that $\tau(A \oplus A) = \tau(A) \oplus \tau(A)$ is discriminated by $\tau(A)$ and hence that $\tau(A)$ is discriminating.

3. An analogous argument to the one above suffices to prove the given assertion.

4. This follows on combining 1 and 2.

5. If A is a finite direct sum of copies of $Z(p^\infty)$ then it has finitely many elements of order p, hence by Proposition 2 (4) A is not discriminating. If A is a direct sum of infinitely many copies of $Z(p^\infty)$ then $A \cong A \oplus A$ and therefore A is discriminating.

6. The image of a divisible subgroup under a homomorphism is again divisible. Since $A \oplus A$ is discriminated by A, it follows that $\delta(A) \oplus \delta(A)$ is discriminated by $\delta(A)$. Consequently $\delta(A)$ is discriminating.

Now a torsion abelian group of finite exponent is a direct sum of cyclic groups (see, e.g., I. Kaplansky [11]). The following lemma is an immediate consequence of this remark and the definition.

Lemma 2 *Let A be a torsion abelian group. Then $\alpha(A, p, k)$ is equal to the cardinality of the set of cyclic summands of order p^k in any decomposition of $A[p^k]$ into a direct sum of cyclic groups.*

We are now in a position to prove our main theorem about discriminating abelian groups.

Theorem A *Let A be a torsion, abelian group and suppose that $\tau_p(D)/\delta(\tau_p(D))$ has no elements of infinite p-height. Then A is discriminating if and only if $\alpha(D, p, k)$ is either zero or infinite for every prime p and every positive integer k and $\beta(D, p)$ is either zero or infinite for every prime p.*

Proof. By Proposition 3, (3), we can assume that A is a p–group.

We begin by proving that if $\alpha(A, p, k) = \alpha$ is non–zero, then it must be infinite. Suppose the contrary, i.e., for some k, α is finite and non–zero. Then if we decompose $A[p^k]$ into a direct sum of cyclic groups, there are exactly α cyclic summands $C(p^k)$ of order p^k. It follows that $A[p^k]$ and therefore A itself, does not contain any direct power of $C(p^k)$ with more then α summands. But $A \oplus A$ has a subgroup which is a direct power of 2α cyclic summands of order p^k. So, by Proposition 2, (1), A is not discriminating. Therefore, if $\alpha(A, p, k) = \alpha$ is non–zero, it must be infinite.

We observe next, that by Proposition 3, (5) and (6), $\beta = \beta(A, p)$ is either 0 or infinite. We consider first the case where $\beta = 0$. So A is a reduced, torsion abelian p–group, without any elements of infinite p-height. If $A = 0$, there is nothing to prove. So we assume that $A \neq 0$. It suffices then to prove that if H is any finite subgroup of $A \oplus A$, then there is a homomorphism of $A \oplus A$ into A which is monic on H. Now every finite subgroup of a p–group P without elements of infinite height can be embedded in a finite direct summand of P (see, e.g., [11]). Therefore we can assume that H is of the form $H = K \oplus K \leq D \oplus D$, where K is a direct summand of A, say $A = K \oplus L$. Suppose that we decompose K into a direct sum of cyclic groups and that a cyclic summand of maximal order has order p^j. Since $\alpha(A, p, j)$ is infinite, there are infinitely many elements of order p^j in L and hence subgroups of L which are arbitrarily large direct powers of $C(p^j)$. Hence there is a homomorphism μ of A into itself which is identical on L and which maps K monomorphically into L. Consider now the homomorphism ρ of $A \oplus A$ into A which maps $A \oplus \{0\}$ identically onto A and maps $\{0\} \oplus A$ into A according to μ. Then ρ maps H monomorphically into A – indeed it maps H onto $K \oplus K$, thought of now as a subgroup of A itself.

We are left with the case where β is infinite. Let H be a finite subgroup of $A \oplus A$. We can assume, without loss of generality, that $H = K \oplus K \leq A \oplus A$. If we now decompose A into a direct sum $A = B \oplus \delta(A)$, then again we can assume that $K = K_1 \oplus A_1 \leq E \oplus \delta(A)$. Since β is infinite, we can find a monomorphism μ of K into $\delta(A)$ such that $\mu(K) \cap D_1 = 0$. It follows that the homomorphism of $A \oplus A$ into A which maps $A \oplus \{0\}$ identically onto A and maps $\{0\} \oplus A$ into A according to μ, is monic on H. This completes the proof of the theorem.

The following examples provide answers to some of the natural questions that might arise in the readers minds.

Example 1 *Let A be the direct sum of a countably infinite number of copies of the groups Z_{p^∞} and a group of order p. Then A is discriminating, but A modulo its maximal divisible subgroup is not.*

Example 2 *Let p be a prime and let A be the unrestricted direct sum of the cyclic groups of order p^i $(i = 1, 2, \ldots)$. Then A is discriminating and therefore so too is its torsion subgroup T. Notice that T is not a direct sum of cyclic groups (see, e.g., [9]).*

Example 3 *Let p be a prime. For a positive integer i denote by D_i the direct sum of a countably infinite number of copies of the cyclic group of order p^i. Then D_i is discriminating by Theorem A. Now, the unrestricted direct sum $A = \overline{\oplus}_i D_i$ of the groups D_i $(i = 1, 2, \ldots)$ is discriminating by Proposition 2. Notice again, that the torsion subgroup of A is not a direct sum of cyclic groups.*

3 Co-discriminating groups

3.1 Co-discrimination and n-separation

Recall that a group H is *co-discriminating* if every group G separating H discriminates H.

We shall have need here of a new notion.

Definition 3 *Let H be a group, let $\mathcal{G}$ be a family of groups and let n be a positive integer. Then we say that $\mathcal{G}$ n–separates H if given any n non-trivial elements $h_1, \ldots, h_n$ of H, there exists a group $G \in \mathcal{G}$ and a homomorphism ϕ of H into G such that $\phi(h_i) \neq 1$ for $i = 1, \ldots, n$.*

Notice, in particular, that if $\mathcal{G}$ n–separates H, then $\mathcal{G}$ certainly separates H. It turns out that, under the right circumstances, if $\mathcal{G}$ 2-separates H, then $\mathcal{G}$ actually discriminates H.

Some of the results that we will prove here are straightforward generalizations of some analogous results of B. Baumslag concerned with groups discriminated by free groups [2].

We recall first the

Definition 4 *A group G is called commutative transitive if commutation is a transitive relation on the set of all non-trivial elements of G, i.e., if a commutes with b and b commutes with c, then a commutes with c, provided a, b, c are non-trivial.*

It is not hard to see that CSA-groups are commutative transitive [16].

The following proposition is useful.

Proposition 4 *Let H be a group and let $\mathcal{G}$ be a family of CSA-groups. Then the following hold:*

1. *Suppose that $\mathcal{G}$ 2-separates H. Then H is a CSA-group.*

2. *If H is commutative transitive and separated by $\mathcal{G}$, then H is a CSA-group.*

Proof. 1. Suppose that H is not a CSA-group. Then there exists a maximal abelian subgroup M of H which is not malnormal. Hence there is a non-trivial element $a \in M$ and an element $x \notin M$ such that $a^x \neq a$ with $a^x = b \in M$. Put $c = [a, x]$. Notice that $c \neq 1$ and that $c \in M$. Now there exists a homomorphism ϕ of H into a group $G \in \mathcal{G}$ such that $\phi(a)$ and $\phi(c)$ are again non-trivial. Let M' be a maximal abelian subgroup of G containing $\phi(M)$. Since G is a CSA-group, $\phi(x) \in M'$. But then $[\phi(a), \phi(x)] = 1$ and therefore $\phi(c) = 1$, a contradiction. This completes the proof of 1.

2. If H is abelian, then H is a CSA-group. Assume then that H is non-abelian. It follows from the assumption that H is commutative transitive, that H has a trivial center. We will prove that $\mathcal{G}$ 2-separates H and then invoke 1. Suppose then that a and b are a pair of non-trivial elements of H. If there exists an element $x \in H$ such that $[a, b^x] \neq 1$, then there exists a homomorphism of H into a group in $\mathcal{G}$ such that the image of $[a, b^x]$ is not equal to 1. Consequently neither the image of a nor that of b is 1, as desired. On the other hand, suppose that a commutes with all of the conjugates of b in H. Then it follows from commutative transitivity that the normal closure of b in H is abelian, i.e., that H contains a non-trivial normal abelian subgroup. Hence we can find a homomorphism of H onto a non-abelian subgroup H' of a group $G \in \mathcal{G}$ which contains a non-trivial abelian normal subgroup M'. Let M be a maximal abelian subgroup of G containing M'. Then M is not malnormal in G, which is impossible since G is a CSA-group. This completes the proof.

Corollary 1 *Let G be a non-abelian CSA-group. Then $G \times G$ is separated by G but not discriminated by G.*

Proof. $G \times G$ is obviously separated by G. If G is non-abelian, then commutation in $G \times G$ is not transitive on the set of all non-trivial elements. Hence, $G \times G$ is not a CSA-group. Thence, by Proposition 4, $G \times G$ can not be discriminated by the CSA-group G. In particular, non-abelian CSA groups are not discriminating.

We remark, in passing, that during the course of the proof of Proposition 4, we have also proved that a non-abelian CSA-group G satisfies the following property: if a and b are non-trivial elements of G, where we allow for the possibility that $a = b$, then there exists an element $x \in G$ such that $[a, b^x] \neq 1$. Such groups are called *groups without zero divisors* or, more frequently, *domains.*

Theorem 1 *Every group without zero divisors is co-discriminating.*

Proof. Let H be a group without zero divisors and let $h_1, \dots, h_n$ be finitely many, non-trivial elements in H. There exists an element $x_1 \in H$ such that $[h_1, h_2^{x_1}] \neq 1$. Hence there exists an element $x_2 \in H$ such that $[[h_1, h_2^{x_1}], h_3^{x_2}] \neq 1$ and so on. It follows that we can find elements $x_1, x_2, \dots, x_{n-1} \in H$ such that

$$c = [\dots[[h_1, h_2^{x_1}], h_3^{x_2}], \dots, h_n^{x_{n-1}}] \neq 1.$$

Now suppose that H is separated by a family of groups $\mathcal{G}$. Let ϕ be a homomorphism of H into a group $G \in \mathcal{G}$ such that $c^\phi \neq 1$. Then ϕ maps each of the elements $h_1, \dots, h_n$ non-trivially into G, as desired.

Corollary 2 *Every non–abelian CSA group is co-discriminating.*

4 Domains

4.1 Domains and products

Proposition 5 *Let A and B be non-trivial groups. Then $A * B$ is a domain unless both A and B are of order 2.*

Proof. Suppose that the order of B is at least 3. Let x and y be non–trivial elements of $A * B$. We claim that there are elements $g, h \in A * B$ such that $[x^g, y^h] \neq 1$. We can assume that x and y are cyclically reduced.

If $\mid x \mid = \mid y \mid = 1$ (here $\mid x \mid$ is the syllable length of x), then $[x, y] \neq 1$ provided x and y lie in different factors, and if they both lie in the same factor, say A, then $[x, y^b] \neq 1$ for any non- trivial $b \in B$.

Suppose now that $\mid x \mid \geq 2$. Then we can assume (conjugating if necessary) that $x = a_1 b_1 \dots a_n b_n$ $(1 \neq a_i \in A,\ 1 \neq b_i \in B)$. Again, conjugating if necessary, we can assume that y is a strictly alternating product of elements from A and B with the first and last elements coming out of B. In this event, the products xy and yx have different syllable length - so x and y do not commute, as desired.

If $\mid A \mid=\mid B \mid= 2$, then $A * B$ has a non-trivial normal abelian subgroup (the derived subgroup). In this event $A * B$ is not a domain.

Notice, that we do not require that the factors A and B in the lemma above be domains. Under the assumptions that A and B are domains, one can prove the following result (see [5] for details).

Theorem 2 *Let A and B be domains. Suppose that C is a subgroup of both A and B satisfying the following condition:*

$$if\ \ c \in C, c \neq 1,\ \ either\ [c, A] \nsubseteq C\ \ or\ \ [c, B] \nsubseteq C.$$

Then the amalgamated free product $H = A *_C B$ is a domain.

Now we are in the position to prove the following generalization of the result of B. Baumslag [2] referred to in the introduction.

Theorem 3 *Let A and B be non-trivial groups not both of order 2. Let G be a group which separates its own free square $G*G$ (in particular a non–abelian torsion-free hyperbolic group). Then the following conditions are equivalent:*

1. *G separates $A * B$;*
2. *G discriminates $A * B$;*
3. *G discriminates each of A and B.*

Proof. By Proposition 5 the group $A * B$ is a domain. So, by Theorem 1, 1 and 2 are equivalent.

3 immediately follows from 2 since A and B are subgroups of $A * B$.

Suppose now that both the groups A and B are discriminated by G. Then in the natural way $A*B$ is discriminated by $G*G$. It is then easy to see that if G separates $G * G$, then, by transitivity of separation, G separates $A * B$.

4.2 One–relator groups

The main theorem of this paper is the following

Theorem B *Let $G = \langle X \mid r = 1\rangle$ be a one-relator group with $\mid X \mid> 2$. Then G is a domain.*

We shall make use of the following lemma.

Lemma 3 *Let N be a domain and let G be a semi–direct product of N and an abelian group. Then G is a domain if and only if G has trivial center.*

Proof. Clearly, if G is a domain then the center of G is trivial.

Suppose now that the center of G is trivial. Observe, that if x and y are a pair of zero divisors in G, then their normal closures $gp_G(x)$ and $gp_G(y)$ in G, commute:

$$[gp_G(x), gp_G(y)] = 1.$$

If $u, v \in G$, then

$$[x, u] \in gp_G(x), \quad [y, v] \in gp_G(y)$$

and hence

$$[gp_G([x, u]), gp_G([y, v])] = 1,$$

Now the center of G is trivial. Therefore there exist elements u and v in G such that $[x, u]$ and $[y, v]$ are non–trivial. But this implies that both $[x, u]$ and $[y, v]$ are zero-divisors in G. Since G/N is abellian, both $[x, u]$ and $[y, v]$ are contained in N. It follows that, in particular, $[x, u]$ is a zero-divisor in N, which contradicts the assumption that N is a domain.

We are now in a position to prove our main theorem. Suppose then that $X = \{t, b, c, \ldots\}$ and that

$$G = \langle X; r = 1\rangle, \tag{1}$$

where r is a cyclically reduced word in X. Let $\sigma_x(r)$ denote the sum of the exponents of all of the occurrences of x in r. We divide the proof into two cases.

Case 1: $\sigma_t(r) = 0$. Denote by N the normal closure in G of $X - t$. Put

$$b_i = t^i b t^{-i}, \; c_i = t^i c t^{-i}, \ldots,$$

Then N is generated by $Y = \{b_i, c_i, \ldots\}$. Since $\sigma_t(r) = 0$, r can be re-expressed as a word $s = s(\ldots, b_j, \ldots, c_k, \ldots)$ in the generators Y, where each of the generators x occuring in r is replaced by x_i, where the subscript i is the sum of the t-exponents of the subword of r preceding x. It follows then, that

$$t^i r t^{-i} = t^i s(\ldots, b_j, \ldots, c_k, \ldots) t^{-i} = s(\ldots, b_{j+i}, \ldots, c_{k+i}, \ldots), \quad (i \in Z).$$

Put

$$s_i = s(\ldots, b_{j+i}, \ldots, c_{k+i}, \ldots), \quad (i \in Z).$$

Using the Reidemeister-Schreier method (see, for example [13]) it is not hard to prove that N has the following presentation:

$$N = \langle b_i, c_i, \ldots \ (i \in Z) \ ; \ s_i = 1 \ (i \in Z) \rangle.$$

Let α and β be correspondingly the minimal and the maximal index of b that occurs in the word s. Now, for arbitrary non-negative integers i, j put

$$N_{-i,j} = \langle c_k, \ldots \ (k \in Z), b_{\alpha-i}, b_{\alpha-i+1} \ldots, b_{\beta+j} \ \mid \ s_{-i} = 1, s_{-i+1} = 1, \ldots, s_j = 1 \rangle.$$

It follows (see [13]) that N is the union of the following ascending chain of subgroups

$$N_{0,0} \leq N_{0,1} \leq N_{-1,1} \leq N_{-1,2} \ldots \leq N_{-i+1,i} \leq N_{-i,i} \leq N_{-i,i+1} \leq \ldots.$$

Notice that in the presentation for $N_{-i,j}$ above, there are infinitely many generators and finitely many relators, so each $N_{-i,j}$ is a free product of two infinite groups. Hence, by Proposition 5, for each $i, j \geq 0$, the group $N_{-i,j}$ is a domain. Since the union of an ascending chain of domains is again a domain, we conclude that N is a domain. Observe that the group G is the semidirect product of N and the infinite cyclic group generated by t. K. Murasugi [15] has proved that every one-relator presentation with more than two generators, defines a group with trivial center. So by Lemma 3, G is a domain.

Case 2: $\sigma_x(r) \neq 0, for\ every\ x \in X.$

Denote by $F = F(X)$ a free group on X. Observe, that $r \notin [F, F]$. Consequently there exists a free basis $Y = \{y_1, y_2, \ldots\}$ of F (see [13] Theorem 3.5), such that

$$r = y_1^e y'$$

where here e is a positive integer and y' is an element in the derived group of F, expressed in terms of the new basis Y of F. Clearly, $\sigma_y(s) = 0$ for every $y \in Y - \{y_1\}$. So if we present G on the generators Y, we find that

$$G =< Y; y_1^e y' > .$$

So we are back to Case 1 and G is a domain, as before.

The condition $\mid X \mid \geq 3$ is essential, since, for example the following one-relator groups are not domains (they contain non-trivial abelian normal subgroups):

$$\langle x, y \mid x^k = y^\ell \rangle, \quad \langle x, y \mid y^{-1}xy = x^\ell \rangle, \quad ((k, \ell \geq 2).$$

It is unclear whether the only one-relator groups which are not domains are those which contain abelian normal subgroups, as in the examples above.

4.3 One-relator groups which are separated by free groups are discriminated by them

The object of this subsection is to prove the

Theorem 6 *Every one-relator group separated by a free group F is discriminated by the free group F.*

Since every one-relator group with at least three generators is, by Theorem B, a domain and hence, by Theorem 1, co-discriminating, it suffices to prove the following lemma.

Lemma 4 *Let G be a group generated by two elements. Then the following conditions are equivalent:*

1. *G is separated by a free group F;*

2. *G is discriminated by a free group F;*

3. *G is either free or free abelian.*

Proof. Notice, that 3 implies 2 (in the case when G is free abelian, it follows from Section **2** that G is discriminated by Z, hence by F). It is obvious that 2 implies 1. So it is enough to prove that 1 implies 33. Observe that every group separated by a torsion-free group is torsion-free; hence G is torsion-free. If G is abelian, then G is free abelian. Suppose now that G is generated by a and b and that $[a, b] \neq 1$. Then there exists a homomorphism $\phi : G \to F$ such that $[\phi(a), \phi(b)] \neq 1$. Therefore the subgroup generated by $\phi(a)$ and $\phi(b)$ is a free group of rank 2; hence ϕ is an isomorphism.

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