Conjugacy separability of non-positively curved groups

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Background and motivation

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Equivalently, G is RF $\iff \bigcap_{N \lhd_f G} N = \{1\}$.

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Recall: $Aut_{pi}(G) = \{\alpha \in Aut(G) \mid \alpha(g) \sim g, \forall\, g \in G\}$ – pointwise inner automorphisms of G.

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- limit groups/fully residually free gps. (Chagas-Zalesskii)
- Let $\mathcal{X}$ be the smallest class of gps. containing virt. free and virt. polyc. gps and closed under cyclic amalgamation. Then each $G \in \mathcal{X}$ is CS (Ribes-Segal-Zalesskii)

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- there is a f.g. group G with a CS sbgp. $H \leqslant G$ s.t. $|G : H| = 2$ and G is not CS (Goryaga)
- there is a f.p. CS group G with a sbgp. $H \leqslant G$ s.t. $|G : H| = 2$ and H is not CS (M.-Martino)

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- If G is RF then the prof. top. on G is induced by the canonical embedding of G in its profinite completion $\widehat{G}$

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Thus, WLOG, we can assume that $x \in X$, where X is a free gen. set of G. I.e., it remains to prove the following

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If $n = 1$, i.e., $y \in X^{\pm 1}$ and $y \neq x$ then y and x can be distinguished in the mod-3 abelianization of G: $M = G/[G, G]G^3$, $|M| < \infty$.

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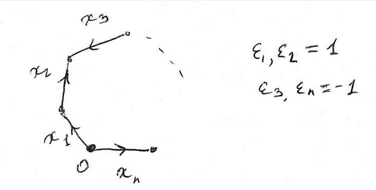

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Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $y \in G \setminus x^G$. WLOG $y = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ is cyclically reduced, where $x_1, \dots x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.
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Right angled Artin groups

Definition

Let Γ be a finite simplicial graph. The right angled Artin group (RAAG) $A(\Gamma)$ is defined by

$$A(\Gamma) = \langle v \in V\Gamma \,\|\, uv = vu \text{ if } (u, v) \in E\Gamma \rangle.$$

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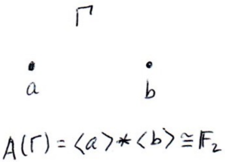

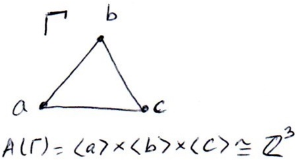

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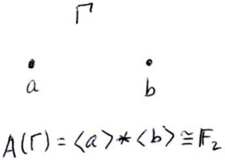

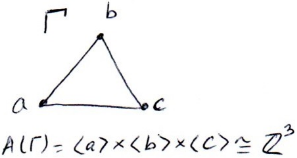

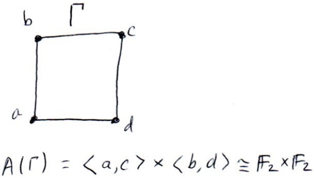

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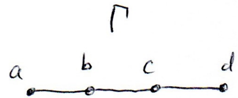

$$A(\Gamma) = \langle a, b, c, d \,\|\, [a,b] = [b,c] = [c,d] = 1 \rangle$$

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The proof is quite long; one of the main ingredients is the decomposition of RAAGs as special HNN-extensions.

Special HNN-extensions

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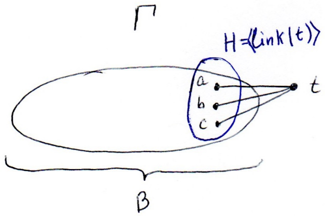

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Remark (Functoriality)

Let A be the spec. HNN-ext. of B w.r.t. $H \leqslant B$. Then every homom. $\phi : B \to L$ extends to a homom. $\hat{\phi} : A \to M$, where M is the spec. HNN-ext. of L w.r.t. $\phi(H)$.

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Problem: to ensure that $\hat{\phi}(x) \not\sim \hat{\phi}(y)$ in M one needs more properties than simply CS of B. That's why we have to prove the stronger result, that A is HCS.

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VCS groups usually act on non-positively curved spaces (e.g., cube complexes).

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But CS may not pass to a finite index overgroup! So, there is still work to be done.

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