Groups elementarily equivalent to a group of upper triangular matrices $T_n(R)$

Mahmood Sohrabi (Stevens Institute)
Joint work with Alexei G. Myasnikov (Stevens Institute)

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(ACC Webinar)

Outline

- I will describe a characterization for groups elementarily equivalent to the group $T_n(R)$ of all invertible upper triangular $n \times n$ matrices, where $n \geq 3$ and R is a characteristic zero integral domain.
- In particular I describe both necessary and sufficient conditions for a group being elementarily equivalent to $T_n(R)$ where R is a characteristic zero algebraically closed field, a real closed field, a number field, or the ring of integers of a number field.

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Elementary theories

Definition

The elementary theory $Th(\mathcal{A})$ of a group $\mathcal{A}$ (or a ring, or an arbitrary structure) in a language L is the set of all first-order sentences in L that are true in $\mathcal{A}$.

Definition

Two groups (rings) $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent in a language L ($\mathcal{A} \equiv \mathcal{B}$) if $Th(\mathcal{A}) = Th(\mathcal{B})$.

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Motivation

Tarski type problems

Given an algebraic structure $\mathfrak{U}$ one can ask if the first-order theory of $\mathfrak{U}$ is decidable, or what are the structures (perhaps under some restrictions) which have the same first-order theory as $\mathfrak{U}$. A. Tarski posed several problems of this nature in the 1950's.

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Indeed, it suffices to mention here results on first-order theories of

- algebraically closed fields, real closed fields (Tarski)
- the fields of p-adic numbers (Ax-Kochen, Ershov),
- abelian groups and modules (Szmielewa, Baur),
- boolean algebras (Tarski, Ershov),
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Some specific Tarski-type problems

Problem 1

Given a classical linear group $G_m(K)$ over a field K, where $G \in \{GL, SL, PGL, PSL,\}$ and $m \geq 2$, characterize all groups elementarily equivalent to $G_m(K)$.

Problem 2

Given a (connected) solvable linear algebraic group G characterize all groups elementarily equivalent to G.

Problem 3

Given an arbitrary polycyclic-by-finite group G characterize all groups elementarily equivalent to G.

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What is already known

- Malcev proved that $G_m(K_1) \equiv G_n(K_2)$ if and only if $m = n$ and $K_1 \equiv K_2$, where K_1, K_2 are fields of characteristic zero.
- In a series of papers Bunina and Mikhalev extended Malcev's results for other rings and groups.
- C. Lasserre and F. Oger (2014) give a criterion for elementary equivalence of two polycyclic groups.
- O. Belegradek (1999) described groups elementarily equivalent to a given nilpotent group $UT_n(\mathbb{Z})$
- Myasnikov-Sohrabi (2011) described all groups elementarily equivalent to a free nilpotent group of finite rank.
- Myasnikov-Sohrabi (2014) developed techniques which seems to be useful in tackling Problem 3 in the nilpotent case.
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Our work's contribution

Our work contributes to the study of the above problems in the follwoing ways.

- First, we present a framework to approach these and similar problems via nilpotent radicals in solvable groups.
- Secondly, we solve these problems for the group of all invertible $n \times n$ upper triangular matrices $T_n(R)$ over a ring R which are model groups for linear solvable groups.
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Abelian extensions and *Ext*

Symmetric 2-cocycles

Assume A and B are abelian groups. A function

$$f : B \times B \to A$$

satisfying

- $f(xy, z)f(x, y) = f(x, yz)f(y, z), \quad \forall x, y, z \in B,$
- $f(1, x) = f(x, 1) = 1, \forall x \in B,$
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The symmetric 2-cocycles form a group $S^2(B, A)$ under point-wise multiplication. A *2-coboundary* $f \in S^2(B, A)$ is a 2-cocycle staisfying:

$$\psi(xy) = f(x, y)\psi(x)\psi(y), \quad \forall x, y \in B,$$

for some function $\psi : B \to A$. Those elements of $S^2(B, A)$ which are coboundaries form a subgroup $B^2(B, A)$.

Abelian extensions

An abelian extension of A by B we mean a short exact sequence of groups

$$1 \to A \xrightarrow{\mu} E \xrightarrow{\nu} B \to 1,$$

where E is abelian.

Fact

There is 1-1 correspondence between the quotient group $Ext(B, A) = S^2(B, A)/B^2(B, A)$ and equivalence classes of abelian extensions of A by B.

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The structure of T_n

- The group $G = T_n(R)$ is a semi-direct product

$$T_n(R) = D_n(R) \ltimes_{\phi_{n,R}} UT_n(R),$$

  where
  - $D_n(R)$ is the subgroup of all diagonal matrices in $T_n(R)$,
  - $UT_n(R)$ denotes the subgroup of all upper unitriangular matrices (i.e. upper triangular with 1's on the diagonal),
  - and the homomorphism $\phi_{n,R} : D_n(R) \to Aut(UT_n(R))$ describes the action of $D_n(R)$ on $UT_n(R)$ by conjugation.
- The subgroup $UT_n(R)$ is the so-called *unipotent radical of* G, i.e. the subgroup consisting of all unipotent matrices in G.
- The subgroup $D_n(R)$ is a direct product $(R^\times)^n$ of n copies of the multiplicative group of units $R^\times$ of R.
- The center $Z(G)$ of G consists of diagonal scalar matrices $Z(G) = \{\alpha I_n : \alpha \in R^\times\} \cong R^\times$, where I_n is the identity matrix.
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- The group $G = T_n(R)$ is a semi-direct product

$$T_n(R) = D_n(R) \ltimes_{\phi_{n,R}} UT_n(R),$$

  where
  - $D_n(R)$ is the subgroup of all diagonal matrices in $T_n(R)$,
  - $UT_n(R)$ denotes the subgroup of all upper unitriangular matrices (i.e. upper triangular with 1's on the diagonal),
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- The subgroup $UT_n(R)$ is the so-called *unipotent radical of* G, i.e. the subgroup consisting of all unipotent matrices in G.
- The subgroup $D_n(R)$ is a direct product $(R^\times)^n$ of n copies of the multiplicative group of units $R^\times$ of R.
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A new structure on T_n

Now we define a new group just by deforming the multiplication on D_n.

- Let $E_n = E_n(R)$ be an arbitrary abelian extension of $Z(G) \cong R^\times$ by $D_n/Z(G) \cong (R^\times)^{n-1}$. As it is customary in extension theory we can assume $E_n = D_n = B \times Z(G)$ as sets (B is complement of $Z(G)$ in D_n), while the product on E_n is defined as follows:

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2 f(x_1, x_2)),$$

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- Next define a map $\psi_{n,R} : E_n \to Aut(UT_n(R))$ by

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Definition of abelian deformations of T_n

Now define a new group structure H on the base set of G by

$$H \stackrel{\text{def}}{=} E_n \ltimes_{\psi_{n,R}} UT_n(R).$$

We call such a group H an *abelian deformation of* $T_n(R)$.

Indeed any abelian extension E_n of $R^\times$ by $(R^\times)^{n-1}$, due to the fact that $Ext((R^\times)^{n-1}, R^\times) \cong \prod_{i=1}^{n-1} Ext(R^\times, R^\times)$, is uniquely determined by some symmetric 2-cocycles $f_i \in S^2(R^\times, R^\times)$, $i = 1, \ldots, n-1$ up to equivalence of extensions. So we denote H by $T_n(R, f_1, \ldots, f_{n-1})$ or $T_n(R, \bar{f})$.

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The main results

Theorem (M.S., A. Myasnikov)

Let $G = T_n(R)$ be the group of invertible $n \times n$ upper triangular matrices over a characteristic zero integral domain R. Then

$$H \equiv G \Rightarrow H \cong T_n(S, f_1, \ldots, f_{n-1}),$$

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Definition of CoT 2-cocycles

Given a ring R as in the statement of the theorem above a symmetric 2-cocycle $f : R^\times \times R^\times \to R^\times$ is said to be *coboundarious on torsion* or *CoT* if the restriction $g : T \times T \to R^\times$, where $T = T(R^\times)$, of f to $T \times T$ is a 2-coboundary.

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Considering the fact that $T(R^\times)$ is finite if R is a number field or the ring of integers of a number field the following result is immediate.

Corollary (M.S., A. Myasnikov)

Assume R is a number field or the ring of integers of a number field. Then $H \equiv T_n(R)$ if and only if $H \cong T_n(S, \bar{f})$ where each f_i is CoT.

In case that R is a characteristic zero algebraically closed field or a real closed field the introduction of abelian deformations is not necessary.

Theorem (M.S., A. Myasnikov)

Assume F is a characteristic zero algebraically closed field or a real closed field. Then

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As for the necessity of introducing abelian deformations we prove the following theorems.

Theorem (M.S., A. Myasnikov)

There is a field K, $K \equiv \mathbb{Q}$ and there are some $f_i \in S^2(K^\times, K^\times)$ such that $T_n(\mathbb{Q}) \equiv T_n(K, \bar{f})$ but $T_n(K, \bar{f}) \not\cong T_n(K')$ for any field K'.

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Assume $\mathcal{O}$ is the ring of integers of an algebraic number field.

1. *If $\mathcal{O}^\times$ is finite, then a group H is elementarily equivalent to $T_n(\mathcal{O})$ if and only if $H \cong T_n(R)$ for some ring $R \equiv \mathcal{O}$.*
2. *If $\mathcal{O}^\times$ is infinite, then there exit $R \equiv \mathcal{O}$ and some $f_i \in S^2(R^\times, R^\times)$ such that $T_n(\mathcal{O}) \equiv T_n(R, \bar{f})$ but $T_n(R, \bar{f}) \not\cong T_n(S)$ for any ring S.*

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How should one approach these problems

To prove the necessity statements ideally one wants to prove that

- $UT_n(R)$ is uniformly definable in G,
- $D_n(R)$ is uniformly definable in G,
- The action of $D_n(R)$ on $UT_n(R)$ can be described using L-formulas

We will see to what extent any of these could be achieved.

Some special elements of $T_n(R) = D_n(R) \ltimes UT_n(R)$

let e_{ij}, $i < j$, be the matrix with ij'th entry 1 and every other entry 0, and let $t_{ij} = I_n + e_{ij}$, where I_n is the $n \times n$ identity matrix. Let also $t_{ij}(\alpha) = I_n + \alpha e_{ij}$, for $\alpha \in R$. These matrices are called transvections and they generate $UT_n(R)$.

Let $diag[\alpha_1, \ldots, \alpha_n]$ be the $n \times n$ diagonal matrix with ii'th entry $\alpha_i \in R^\times$. The group $D_n(R)$ consists of these elements as the α_i range over $R^\times$. Now consider the following diagonal matrices

$$d_i(\alpha) \stackrel{\text{def}}{=} diag[1, \ldots, \underbrace{\alpha}_{i'\text{th}}, \ldots, 1],$$

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Clearly the $d_i(\alpha)$ generate $D_n(F)$ as α ranges over $R^\times$.

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Recovering the unipotent radical

Fitting Subgroups

By the *Fitting subgroup* of a group G, denoted by $Fitt(G)$, we mean the subgroup generated by all normal nilpotent subgroups of G.

Denote by $\mathcal{P}$ the class of groups G where the Fitting subgroup is itself nilpotent. For example every polycyclic-by-finite group is in $\mathcal{P}$. Also $T_n(R)$ for any commutative associative ring R with unit is in $\mathcal{P}$.

Lemma (A. Myasnikov, V. Romankov, M.S.)

Assume G is a group in $\mathcal{P}$. There is a formula that defines $Fitt(G)$ in G uniformly with respect to $Th(G)$. In particular the class $\mathcal{P}$ is an elementary class.

Lemma

The derived subgroup of G' of $G = T_n(R)$ is definable in G.

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Lemma

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Indeed $UT_n(R)$ is not necessarily a definable subgroup of G. However we can prove the following

Lemma

Assume $G = T_n(R)$ and $H \equiv G$. Then $Z(H)$ contains a unique element of order 2 denoted by $-I_n$. Therefore the subgroup $UT_n(R) \times \{\pm I_n\}$ is definable in G.

There is a problem though. In general for $G = T_n(R)$ where R is a characteristic zero integral domain

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Consider the group $N = UT_n(R)$, where R is a commutative associative ring with unit. Then for each $1 \leq i < j \leq n$ the one-parameter subgroups $T_{ij} = \{t_{ij}(\alpha) : \alpha \in R\}$ are definable in N, unless $j = i + 1$. If $j = i + 1$ then the subgroup $C_{ij} = T_{ij} \cdot Z(N)$ is definable in N. The $T_{i,i+1}$ are not in general definable in N.

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One can improve Belegradek's lemma as follows:

Lemma

The R-module structure of each T_{ij} is interpretable in N if $j > i+1$. If $j = i+1$ then the R-module $C_{i,i+1}$ is interpretable in N.

The good thing is the following fact:

$$[d_2, \pm C_{12}] = \langle t_{12}(2\alpha) : \alpha \in R \rangle,$$

which is close enough to T_{12}.
So indeed using these first-order equations we can recover all the $\pm T_{i,i+1}$.
With a little bit of effort we can prove now that

Proposition

If $H \equiv T_n(R)$ for a characteristic zero integral domain R then there exists a ring $S \equiv R$ and an abelian subgroup $E_n \equiv D_n(R)$ of H such that

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If we can encode the fact that $D_n \cong (R^\times)^n$ then we are basically done. But all we know are the following first-order equations:

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By the way, D_n is defined in G as the centralizer $C_G(\{d_i : i = 1, \ldots n\})$.

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We can coordinatize D_n using the equations above but everything will be modulo the center. So we have proved that

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Let $G = T_n(R)$, and H is a group $H \equiv G$. Then

(a) *For each $1 \leq i \leq n$ the subgroup $\Delta_i(R) \stackrel{def}{=} d_i(R^\times) \cdot Z(G)$ is first-order definable in $D_n = D_n(R)$ by an L-formula. Moreover there exists a ring $S \equiv R$ and for each $i = 1, \ldots, n$ a subgroup $\Lambda_i < E_n$ of H defined in H by the same formula that defines Δ_i in D_n such that $\Lambda_i / Z(H) \cong S^\times$.*

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$H \equiv (T_n(R) = D_n(R) \ltimes_{\phi_{n,R}} UT_n(R))$ *then*

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where $S \equiv R$ *and* E_n *is an abelian extension of* $Z(H) \cong S^\times$ *by* $(S^\times)^{n-1}$ *and* $\psi_{n,s} \approx \phi_{n,s}$.

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The sufficient conditions

- Question: Are all $T_n(S, \bar{f})$ elementarily equivalent to $T_n(R)$ if $R \equiv S$?
- Answer: No!

- Recall that for ring R a 2-cocycle $f : R^\times \times R^\times \to R^\times$ is said to be CoT if the restriction $g : T \times T \to R^\times$, where $T = T(R^\times)$ is the torsion subgroup of $R^\times$, of f to $T \times T$ is a 2-coboundary.
- Assume A is abelian extension of $A_1 = R^\times$ by $A_2 = R^\times$, and T_2 is the copy of T in A_2. Then f is CoT if and only if the subgroup H of A generated by A_1 and any preimage of T_2 in A splits over A_1, i.e. $H \cong A_1 \times T_2$.
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Assume R is a characteristic zero integral domain so that the maximal torsion subgroup of $R^\times$ is finite. Assume $f \in S^2(R^\times, R^\times)$ is CoT and $(I, \mathcal{D})$ is an ultra-filter so that ultraproduct $(R^\times)^*$ of $R^\times$ over $\mathcal{D}$ is $\aleph_1$-saturated. Then the 2-cocycle $f^* \in S^2((R^\times)^*, (R^\times)^*)$ induced by f is a 2-coboundary.

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Are abelian deformations necessary after all?

Proposition

Assume R and S are characteristic zero integral domains with unit. Let $\phi : T_n(R, \bar{f}) \to T_n(S)$ be an isomorphism of abstract groups. Then $R \cong S$ as rings and all the symmetric 2-cocycles f_i are 2-coboundaries.

Proposition

For any ring $\mathcal{O}$ of integers with infinite $\mathcal{O}^\times$ of a number field F there exists a ring $S \equiv \mathcal{O}$ where $Ext(S^\times, S^\times) \neq 1$.

Using Romanovskii-Robinson we can prove that $\lambda^{\mathbb{Z}} \leq \mathcal{O}^\times$, where λ is any element of a definable subgroup B of finite index in $\mathcal{O}^\times$, with the corresponding ring structure is interpretable in $\mathcal{O}^\times \ltimes \mathcal{O}$ and thus in $T_n(\mathcal{O})$.

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Theorem

Assume $\mathcal{O}$ is the ring of integers of an algebraic number field.

1. *If $\mathcal{O}^\times$ is finite, then*
$$H \equiv T_n(\mathcal{O}) \Leftrightarrow H \cong T_n(R)$$
for some ring $R \equiv \mathcal{O}$.

2. *If $\mathcal{O}^\times$ is infinite, then there exit $R \equiv \mathcal{O}$ and some $f_i \in S^2(R^\times, R^\times)$ such that*
$$T_n(\mathcal{O}) \equiv T_n(R, \bar{f})$$
but
$$T_n(R, \bar{f}) \ncong T_n(S)$$
for any ring S.

Our work's contribution in studying Problems 1-3

Our work contributes to the study of Problems 1-3 in the follwoing ways:

- First, we present a framework to approach these and similar problems via nilpotent radicals in solvable groups.
- Secondly, we solve these problems for the group of all invertible $n \times n$ upper triangular matrices $T_n(R)$ over a ring R which are model groups for linear solvable groups
- The groups $T_n(R)$, as they are, play an important part in the study of model theory of groups $G_m(K)$ from Problem 1.

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