Algorithmic questions for torsion-free hyperbolic groups Γ and for Γ-limit groups

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Overview

This is a joint talk with A. Taam and joint results with A. Miasnikov

Background

- The study of f.g. fully residually free groups (limit groups) was motivated, in part, by the study of the elementary theory of free groups, and resulted in positive answers to fundamental Tarski questions for free groups (Kharlampovich-Myasnikov, Sela).

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- algorithms for certain canonical (JSJ) decompositions of limit groups were central in those works, by giving canonical embeddings and description of homomorphisms and automorphisms
- throughout this talk, unless stated otherwise, let Γ be a fixed torsion-free non-elementary hyperbolic group, with a finite generating set A.

fully residually G groups

let G be a group (class of groups)

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Definition

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- If G is equationally Noetherian (every system of equations in n variables is equivalent to a finite subsystem) these definitions are equivalent for finitely generated groups (we are usually interested in the cases of $G = F$ a free group, $G = \Gamma$ a torsion-free hyperbolic group, or $G = \mathcal{G}$ a toral relatively hyperbolic group, all of which are equationally Noetherian).

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- in '05 Champetier and Guirardel showed that f.g. fully residually free is equivalent to F-limit groups.

Γ-limit groups

Characterization Theorem

Let Γ be an equationally Noetherian group and G a finitely generated group containing Γ. Then the following conditions are equivalent:

1) G is fully residually Γ;
2) G is universally equivalent to Γ (in the language with constants);
3) G is the coordinate group of an irreducible algebraic set over Γ;
4) G is a Γ- limit group;
5) G embeds into an ultrapower of Γ.

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- Let $R(S) = \{T(X, A) \in G[X] | \forall Z \in G^n (S(Z, A) = 1 \to T(Z, A) = 1)\}$. We call $G_{R(S)} = G[X]/R(S)$ the coordinate group of S (over G). Every solution of $S(X, A) = 1$ in G corresponds to a G-homomorphism $G_{R(S)} \to G$.

NTQ groups

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Definition

A system of equations $S(X, A) = 1$ over a group G, is called *triangular quasi-quadratic over* G or G-TQ, if it can be partitioned into subsystems: $S_i(X_i, C_i) = 1; 1 \leq i \leq n$ where $\{X_1, \ldots, X_n\}$ is a partition of X, and setting $G_i = G[X_i, \ldots, X_n, T]/R_G(S_i, \ldots, S_n)$ for $1 \leq i \leq n$ and $G_{n+1} = G * F(T)$, we have $C_i = X_{i+1} \cup \ldots \cup X_n \cup A \subset G_{i+1}$ for $1 \leq i \leq n-1$ and $C_n = A$. The number n is called the *depth* of the system. Furthermore, for each i the subsystems S_i must have one of the following forms:

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(IV) S_i is empty

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- $S(X, A) = 1$ is called *non-degenerate triangular quasi-quadratic over* G or G-NTQ if it is G-TQ and for every i, the system $S_i(X_i, C_i) = 1$ has a solution in G_{i+1}, and if S_i is of form (II) the set U generates a centralizer in G_{i+1}.

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- A *regular* G-NTQ system is a G-NTQ system in which each non-empty quadratic equation S_i is in standard form, and either $\chi(S_i) \leq -2$ and the quadratic equation has a non-commutative solution in G_{i+1}, or it is an equation of the form $[x, y]d = 1$ or $[x_1, y_1][x_2, y_2] = 1$.

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- if G is toral relatively hyperbolic, every G-NTQ group is also toral relatively hyperbolic

Canonical representatives

- There are some particularly useful tools for working with Γ-limit groups (hyperbolic groups in general) and equations over groups which make it possible to take advantage of results in free groups.

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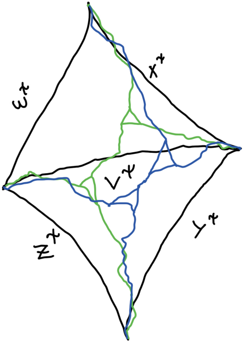
- Note that Dahmani and Groves construct canonical representatives for toral relatively hyperbolic groups in free products of free groups and free abelian groups.

Canonical representatives

$$wxyz = 1 \to wxv = 1, v^{-1}yz = 1$$



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Theorem (Kharlampovich, Macdonald 2013)

Given a system $S(Z, A) = 1$ over Γ, there is a finite tree $\mathcal{T}$, where every branch b_i corresponds to a Γ-NTQ group N_i and homomorphism $\phi_i : \Gamma_{R(S)} \to N_i$, where each $H_i = \phi_i(\Gamma_{R(S)})$ is a Γ-limit group, and for any homomorphism $\psi : \Gamma_{R(S)} \to \Gamma$, there is a homomorphism $\pi_i : H_i \to \Gamma$, for some i, such that $\psi = \phi_i \pi_i$.

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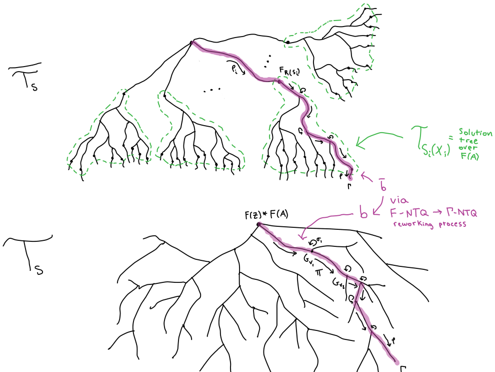
- The tree is constructed using canonical representatives to get all solutions of the system in Γ from solutions of a finite number of systems of equations over a free $F(A)$. Then $F - NTQ$ groups can be reworked to give Γ-NTQ groups.
- There are finitely many (isomorphism classes) of maximal (w.r.t just quotient ordering) Γ-limit quotients The collection $\{H_i\}$ contains a representative of each isomorphism class.

F-NTQ to Γ-NTQ reworking process



F-NTQ to Γ-NTQ reworking process

The reworking process is necessary since considering each F-NTQ system as a system over Γ gives groups through which solutions factor, but may no longer be Γ-NTQ groups, since relators of Γ may kill certain parts of the NTQ structure. The process gives explict constructions for how to add new variables and relators depending on the form of each equation. Finally the resulting system is shown to be equivalent to a Γ-NTQ one.

Splittings of groups

- Recall that a graph of groups is a connected graph $X(V, E)$ with a group G_v for each vertex $v \in V$, and a group G_e with monomorphisms $\alpha_e : G_e \to G_{\partial_0(e)}$, $\beta_e : G_e \to G_{\partial_1(e)}$ for each $e \in E$.

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- A splitting of a group G over some class of group $\mathcal{E}$ is an isomorphism from G to π of a graph of groups with all edge groups in $\mathcal{E}$.

Classification of vertex groups

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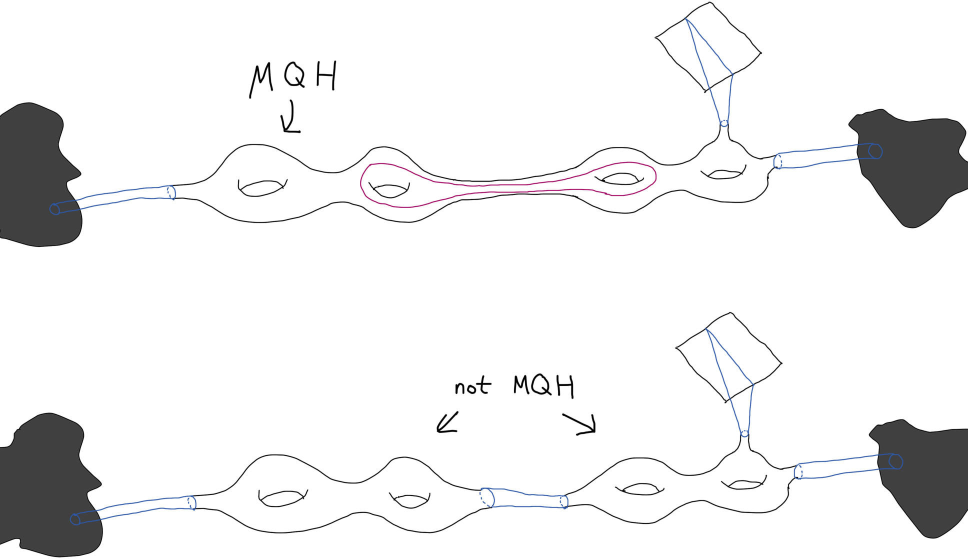
- A non-QH, non-abelian vertex group is called a rigid subgroup.

Classification of vertex groups



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- FACT: the edge groups of 2 elementary splittings (1 edge) are always both hyperbolic, or both elliptic w.r.t. to the other splitting.
- A reduced (image of each edge group is a proper subgroup of its vertex group) abelian splitting is essential if for any $g \in G$ with $g^k \in G_e$ for some k, then $g \in G_e$. An essential splitting of G is primary if each noncyclic abelian subgroup of G can be conjugated into one of its vertex groups.

JSJ for Γ-limits - definition

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(i) Every MQH subgroup of H is elliptic in D, and every vertex group of D which is not conjugate to a MQH subgroup, is elliptic in any splitting.

(ii) Any elementary splitting which is hyperbolic in another elementary splitting, can be obtained from D by cutting along a weakly essential s.c.c., a 2-orbifold corresponding to a MQH subgroup.

JSJ for Γ-limits - definition

Let H be a freely indecomposable Γ-limit group. By Sela ('09), there exists is a primary, reduced, unfolded splitting D of H, called an primary abelian JSJ decomposition, where:

(i) Every MQH subgroup of H is elliptic in D, and every vertex group of D which is not conjugate to a MQH subgroup, is elliptic in any splitting.

(ii) Any elementary splitting which is hyperbolic in another elementary splitting, can be obtained from D by cutting along a weakly essential s.c.c., a 2-orbifold corresponding to a MQH subgroup.

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(v) All non-cyclic abelian subgroups are elliptic.

Canonical NTQ systems

Definition

An NTQ system is canonical for a group $\Gamma_{R(S)}$ if a quadratic system of equations on each level corresponds to the JSJ

Results

Theorem

(Kh., Miasnikov, A. Taam) Let $S(Z, A) = 1$ be a finite system of equations over Γ. There is an algorithm to construct a complete set of canonical NTQ systems for $\Gamma_{R(S)}$.
Moreover, there is an algorithm to construct a complete set of canonical NTQ systems for each maximal Γ-limit quotient of $\Gamma_{R(S)}$.

Difficulties

- Γ-limit groups are more difficult, especially algorithmically, as they are not necessarily finitely presented (since subgroups of Γ are not necessarily finitely presented, e.g. non-quasiconvex).

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- How Γ-limit groups are given is significant. We are interested in describing the Γ-limit quotients $\{H_i\}$ obtained from the tree $\mathcal{T}$ constructed by Kharlampovich and Macdonald, described earlier.

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For each of the Γ-limit quotients H_i

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- from this, we can find the maximal Γ-limit quotients relative to conjugates of those subgroups of Γ
- extending centralizers of edge groups for these presentations we can algorithmically construct canonical Γ- NTQ systems.