Branch groups: groups that look like trees

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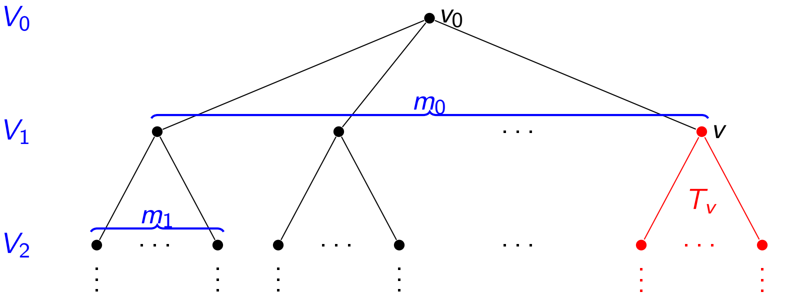
Outline

Trees



Definition

$(m_n)_{n\geq 0}$ sequence of integers ≥ 2.
T is a rooted tree of type $(m_n)_n$ if T is a tree with root v_0 of degree m_0 s.t. every vertex at distance $n \geq 1$ from v_0 has degree $m_n + 1$.

V_n = vertices at distance n from root
T_v is subtree rooted at v

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  - Non-uniform exponential word growth (Wilson)
  - Amenable but not elementary amenable groups (Grigorchuk)
  - Filling gaps in subgroup growth spectrum (Segal)

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  Regular trees are self-similar/fractal. Many of these groups are also "self-similar". Self-similar groups (=groups generated by automata) appear naturally as iterated monodromy groups of self-coverings of topological spaces and encode combinatorial information about the dynamics of these coverings (Nekrashevych).

Example: Gupta–Sidki p-groups

$T = T(p)$, p =odd prime

$a := (1\,2\,\ldots\,p)$ on V_1
$b := (a, a^{-1}, 1, \ldots, 1, b)$.

$G := \langle a, b \rangle$

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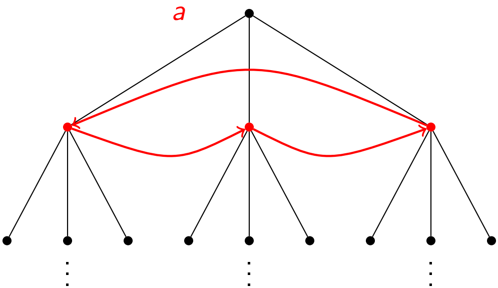

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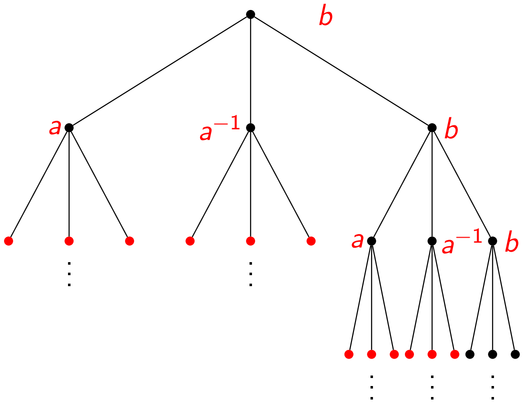

Outline

1. Introduction
2. Self-similarity
3. Branch structure

Projections

Definition

For G acting faithfully on T:

$\mathrm{St}_G(v) := \{g \in G : vg = v\}$ is the stabilizer of v;

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Definition

$G_v := \varphi_v(\mathrm{St}_G(v))$ is the vertex section/projection of G at v.

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Question: Is there a f.p. branch/self-similar group?

More self-similar results

Take this even further:

Theorem (G, 2013)

Let G be the Gupta–Sidki 3-group. If $H \leq G$ is finitely generated and infinite then there exists $v \in T$ with $H_v = G$.

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Cfr:

Theorem (Grigorchuk–Wilson, 2001)

All infinite finitely generated subgroups of the Grigorchuk group Γ are commensurable with Γ.

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The "technical work" only works for $p = 3$; everything else works for all odd primes.

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Outline

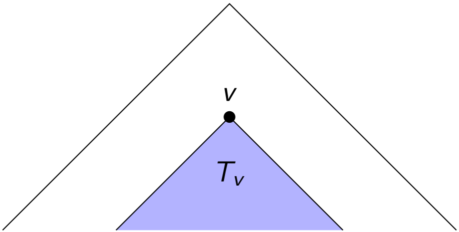
Branch group: definition

T =rooted tree of type $(m_n)_n$. G acts faithfully on T.

Definition

$\mathrm{rst}_G(v) := \{g \in G : g \text{ fixes all vertices outside } T_v\}$ is the rigid stabilizer of $v \in T$.

$\mathrm{rst}_G(n) := \prod_{v \in V_n} \mathrm{rst}_G(v)$ is the rigid stabilizer of level n.



Branch group: definition

Definition

G acts as a branch group on T iff for every n:

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Examples

- For all n, $A = Aut(T)$ acts transitively on V_n with kernel $\mathrm{rst}_A(n)$.
- Gupta–Sidki p-groups
- Grigorchuk groups
- Aleshin group

Subgroups of branch groups

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We can use this to give an isomorphism invariant for H:

Finite index subgroups of branch groups

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Say $V_n = X_1 \sqcup \ldots \sqcup X_r$, each X_i an H-orbit.

Then $\mathrm{rst}_G(X_i)' \triangleleft H$ and $\mathrm{rst}_G(n)' = \prod \mathrm{rst}_G(X_i)' \triangleleft H$.

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$b(H) =$ maximum number of orbits of H on any layer of T.

How it all fits together

$b(H)$ behaves well under direct products

Let $H \leq_f H_1 \times \ldots \times H_r$ be subdirect; $b(H_i)$ finite.
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If Γ_1 and Γ_2 are commensurable, then $n_1 \equiv n_2 \mod p-1$.

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So the Gupta–Sidki 3-group has 3 commensurability classes of f.g. subgroups.

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Turns out we only need to look at subgroups with finitely many conjugates.

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Definition

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Conjugation by G induces a well-defined action of G on $\mathcal{L}$.
So, reformulating, we have

Theorem 2

Every element of $\mathcal{L}$ has as a representative some $\mathrm{rst}(X)$ where X is an H-orbit for some $H \leq_f G$.

Application: Congruence subgroup property

By analogy with the classical case of linear algebraic groups, we have

Definition

A group G acting faithfully on a rooted tree has the congruence subgroup property (CSP) if for every $H \leq_f G$ there is some n with $\mathrm{St}(n) \leq H$.

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Theorem 3 (G, 2014)

Whether a branch group has CSP or not is independent of the branch action.

Proof ingredients

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Lemma

If G acts as a branch group on T then T embeds G-equivariantly in $\mathcal{L}$: $v \mapsto [\mathrm{rst}_G(v)]$.

Proof

To show that having CSP is independent of the branch action, we need to show that given two branch actions $\sigma : G \to \mathrm{Aut}(T_\sigma)$ and $\rho : G \to \mathrm{Aut}(T_\rho)$ every $\mathrm{St}_\sigma(n)$ contains some $\mathrm{St}_\rho(m)$ and vice-versa.

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- $x \in \bigcap_{g\in G} \mathrm{St}_\sigma(ug) = \mathrm{St}_\sigma(n)$.

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Q How many "different" branch actions can a given group have? On what trees?

Thank you for your attention :)