Compactness properties of systems of equations

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"If God does not exist, everything is permitted"
F. Dostoevsky

"If all groups are equationally Noetherian, everything is permitted"
The sense of this talk

Plan

We consider infinite systems of equations and their finite subsystems.

1. Equationally Noetherian groups and semigroups;
2. weakly Equationally Noetherian groups and semigroups;
3. $\mathbf{q}_\omega$-compact groups and semigroups.

Below all infinite systems depend on a finite set of variables.

Group and semigroup equations

Let G be a group and $X = \{x_1, x_2, \ldots, x_n\}$ a set of variables. An expression $w(X) = 1$, $w(X) \in F(X) * G$ ($F(X)$ is the free group generated by X) is called an equation over a group G. For example,

$$[x_1, g] = 1, \; g_1^{-1} x_1 g_1 x_2^{-1} = 1.$$

Let S be a semigroup and $X = \{x_1, x_2, \ldots, x_n\}$ a set of variables. An expression $u(X) = w(X)$, (where $u(X), w(X)$ are products of variables X and elements of S) is called an equation over a semigroup S. For example,

$$s_1 x_1^2 s_2 = s_3 x_2^3, \; x_1 x_2 = x_3 s_1 x_2^2.$$

System of equations = system (for shortness). Let $\mathrm{V}(\mathbf{S})$ denote the solution set of $\mathbf{S}$.

Two systems are equivalent if they have the same solution set over a given group (semigroup).

Noetherian property = atheism in religion

A group (semigroup) is equationally Noetherian if any system of equations $\mathbf{S}$ is equivalent to a finite subsystem. The class of equationally Noetherian groups (semigroups) is denoted by $\mathbf{N}$

Examples of equationally Noetherian groups and semigroups

Free, linear, commutative groups and semigroups. And other examples.

How to find non-equationally Noetherian group (semigroup)?

By the definition in such group (semigroup) there exists an infinite chain of algebraic sets:

$$Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_n \supseteq \ldots$$

De facto, there are three popular approaches:

1. chain of homomorphic kernels;
2. linear ordered idempotents (it works only for semigroups);
3. chain of centralizers.

Infinite chain of homomorphic kernels

Let α_i be an infinite set of endomorphisms of a finitely generated group G, and the kernels satisfy

$$ker(\alpha_1) \subset ker(\alpha_2) \subset \dots$$

Let $w_i \in ker(\alpha_{i+1}) \setminus ker(\alpha_i)$. Since G is finitely generated, all w_i depend on a finite set of generators $g_1, g_2, \dots, g_n$.
It is directly checked that the system $\mathbf{S} = \{w_i(X) = 1\}$ is not equivalent to any finite subsystem (defining $\mathbf{S}$, we replace each g_j to a variable x_j in $w_i \in \mathbf{S}$).

Baumslag-Solitar group

Theorem[Harju, Karhumaki, Plandowski]

If a finitely generated group G is non-hopfian, G is not equationally Noetherian.

Sketch of the proof: epimorphisms with nontrivial kernel generate the chain of homomorphic kernels.

Corollary[Harju, Karhumaki, Plandowski]

The Bauslag-Solitar group is not equationally Neotherian. Moreover, there exists a system $\mathbf{S}$ with no constants which is not equivalent to its finite subsystems.

Infinite chain of idempotents. Bicyclic semigroup

Consider a monoid

$$B = \langle a, b | ab = 1 \rangle.$$

which is called the bicyclic semigroup.
Properties of B:

1. any element of B is written in the normal form $b^n a^m$ $(n, m \in \mathbb{N})$;
2. elements $b^n a^n$ are idempotents:

$$b^n a^n b^n a^n = b^n (a^n b^n) a^n = b^n a^n;$$

3. the idempotents form a chain

$$1 > ba > b^2 a^2 > \dots$$

relative to the order

$$x \leq y \Leftrightarrow xy = x.$$

B is not equationally Noetherian

Proof [Harju, Karhumaki, Plandowski]

Consider a system

$$\begin{cases} x_1x_2x_3 = x_3, \\ x_1^2x_2^2x_3 = x_3, \\ x_1^3x_2^3x_3 = x_3, \\ \dots \end{cases}$$

A point $(b, a, b^n a^n)$ is a solution of the first n equations:

$$b^i a^i b^n a^n = b^i b^{n-i} a^n = b^n a^n,$$

but it does not satisfy the $n+1$-th equation

$$b^{n+1} a^{n+1} b^n a^n = b^i a a^n = b^{n+1} a^{n+1} \neq b^n a^n.$$

Thus, $B \notin \mathbf{N}$.

Chain of centralizers

Theorem [Gupta, Romanovskiy]

Define a partially commutative group of the nilpotency class 2:

$$GR = \langle a_1, a_2, \dots, b_1, b_2, \dots | [b_i, a_j] = 1 \rangle$$

for all $j \le i$.
The group GR is not equationally Noetherian.

Proof. Consider a system

$$[x, a_1] = 1, \ [x, a_2] = 1, \ \dots$$

it corresponds to the chain of centralizers

$$C(a_1) \supseteq C(a_2) \supseteq \dots$$

Consistently Noetherian groups and semigroups

By the definition, any inconsistent system $\mathbf{S}$ over an equationally Noetherian group (semigroup) contains an inconsistent finite subsystem.

Define $S \in \mathbf{N}_c$ if any consistent system of equations $\mathbf{S}$ is equivalent over S to its finite subsystem. By the definition, $\mathbf{N} \subseteq \mathbf{N}_c$

Problem

$\mathbf{N} = \mathbf{N}_c$?

First, we negatively solve this problem for semigroups.
To disprove the equality one should find a semigroup S such that

Any consistent system $\mathbf{S}$ is equivalent to its finite subsystem over S.
There is an inconsistent system $\mathbf{S}_0$ whose finite subsystems are consistent over S.

Let $\mathcal{F}$ be a free semilattice of infinite rank. $\mathcal{F}$ is isomorphic to the class of all finite subsets of $\{a_1, a_2, \ldots, a_n, \ldots\}$ relative to the union operation.

$$x_1x_2a_1 = x_1x_3a_2a_3,\ x_1 = x_2x_3a_1$$

are the examples of equations over $\mathcal{F}$.

Theorem

$\mathcal{F} \in \mathbf{N}_c \setminus \mathbf{N}$. It follows that any infinite subsemigroup of $\mathcal{F} \in \mathbf{N}_c \setminus \mathbf{N}$.

Are there $\mathbf{N}_c$-groups?

Open problem

Is there a group $G \in \mathbf{N} \setminus \mathbf{N}_c$?

The problem is not simple, since each of the approaches above does not work!
Using chains of homomorphic kernels or centralizers, one can obtain only consistent system!!!

In the paper by Baumslag, Miasnikov, Romankov "Two theorems about equationally noetherian groups" if was defined a group G and an inconsistent system $\mathbf{S}$ such that any finite subsystem of $\mathbf{S}$ is consistent. However, G is very complicated, and it is very hard to prove $G \in \mathbf{N}_c$.

$\mathbf{q}_\omega$-compact groups

A group G is $\mathbf{q}_\omega$-compact if for any system $\mathbf{S}$ and equation $w(X)=1$ such that

$$\mathrm{V}_G(\mathbf{S}) \subseteq \mathrm{V}_G(w(X)=1)$$

there exists a finite subsystem $\mathbf{S}' \subseteq \mathbf{S}$ with

$$\mathrm{V}_G(\mathbf{S}') \subseteq \mathrm{V}_G(w(X)=1).$$

Obviously, any equationally Noetherian group is $\mathbf{q}_\omega$-compact.
Denote by $\mathbf{Q}$ the class of all $\mathbf{q}_\omega$-compact groups.

Plotkin's monster

Let PM be the Cartesian product of all finitely generated groups (Plotkin's Monster).

Theorem [B.Plotkin]

$PM \in \mathbf{Q} \setminus \mathbf{N}$.

$PM \notin \mathbf{N}$ follows from the embedding of Bauslag-Solitar group into PM.
$PM \in \mathbf{Q}$ follows from theorem of universal algebra.

Problem [Daniyarova, Miasnikov, Remeslennikov]

Is there a simple example of $\mathbf{q}_\omega$-compact group?

Negative example

Theorem

The group GR is not $\mathbf{q}_\omega$-compact.

Consider a system

$$\mathbf{S} = \{[x, a_i] = 1 | i \in \mathbb{N}\} \cup \{y = y\}.$$

x belongs to the center of the group. Therefore, the equation $[x, y] = 1$ follows from $\mathbf{S}$, and moreover

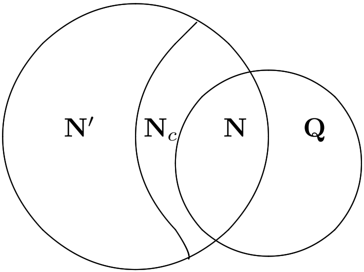
$$\mathrm{V}(\mathbf{S}) \subseteq \mathrm{V}([x, y] = 1).$$

There is not a finite subsystem $\mathbf{S}' \subseteq \mathbf{S}$ satisfying the inclusion above.

Weakly equationally Noetherian groups and semigroups

A group G (semigroup S) is weakly equationally Noetherian if any system $\mathbf{S}$ is equivalent over G (resp., S) to some finite system.
Obviously, $\mathbf{N} \subseteq \mathbf{N}'$.
Denote the class of weakly equationally Noetherian groups (semigroups) by $\mathbf{N}'$.

Intersection of $\mathbf{Q}$ and $\mathbf{N}'$



Amazingly, $\mathbf{N}' \cap \mathbf{Q} = \mathbf{N}$
What class ($\mathbf{Q}$ or $\mathbf{N}'$) is simpler? I think $\mathbf{N}'$. Why?

All famous semigroups belong to $\mathbf{N}'$

1. free commutative idempotent semigroups of infinite rank (precisely, they belong to $\mathbf{N}_c \subseteq \mathbf{N}'$);
2. free left regular idempotent semigroups (they belong to $\mathbf{N}' \setminus \mathbf{N}_c$);
3. however, I do not know any group from $\mathbf{N}' \setminus \mathbf{N}$.

Really, it is hard to construct $\mathbf{q}_\omega$-compact semigroup!

We know that $B \notin \mathbf{N}$. We can pose the problem.

Problem

Prove $B \in \mathbf{Q}$.

This problem is a generalization of the famous one:

Famous "problem" of semigroup theory

Prove something positive for the bicyclic semigroup B.

Really, it is hard to construct $\mathbf{q}_\omega$-compact group!

First, your group should not be equationally Noetherian. Suppose there exists an infinite chain of centralizers

$$C(g_1) \supseteq C(g_1) \supseteq \ldots \supseteq Z(G)$$

Therefore, the system $\mathbf{S} = \{[x, g_i] = 1, y = y\}$ is not equivalent to its finite subsystems.
However, we obtain a problem: does the equation $[x, y] = 1$ follows from $\mathbf{S}$ $(\mathrm{V}_G(\mathbf{S}) \subseteq \mathrm{V}([x, y] = 1))$? If does, we obtain a contradiction with the definition of $\mathbf{q}_\omega$-compactness. Thus, the set $\mathrm{V}_G(\mathbf{S})$ should contain non-central (transcendent) elements.

The group

$$GR_q = \langle a_1, a_2, \ldots, b_1, b_2, \ldots, c_1, c_2 | [b_i, a_j] = 1; c_1, c_2 \text{ commute with all } a_i \rangle$$

is $\mathbf{q}_\omega$-compact for systems with no occurrences of c_1, c_2.
The red condition is essential, since

$$\mathbf{S} = \{[x, a_i] = [x, c_j] = 1\},\ \mathrm{V}(\mathbf{S}) = Z(GR_q).$$

Thus,

$$\mathrm{V}(\mathbf{S}) \subseteq \mathrm{V}([x, y] = 1),$$

and there is not a finite subsystem satisfying the inclusion above.

Thank you for attention!