On some algorithmic problems in nilpotent groups (joint w. François Dahmani)

Nicholas Touikan

Stevens Institute of Technology

October 16 2014
ACC Webinar

The paper

For details see section 6 of the paper **Isomorphisms using Dehn fillings: the splitting case** found at

`http://arxiv.org/abs/1311.3937`

Separating torsion

Let G be a group and let $H \lhd_c G$ be a finite index characteristic subgroup, such that every finite order $\alpha \in \mathrm{Out}\,(G)$ survives in the homomorphism

$$\mathrm{Out}\,(G) \to \mathrm{Out}\,(G/H), \tag{1}$$

Then we say that H *separates torsion in* $\mathrm{Out}\,(G)$.

Separating torsion

Let G be a group and let $H \lhd_c G$ be a finite index characteristic subgroup, such that every finite order $\alpha \in \mathrm{Out}\,(G)$ survives in the homomorphism

$$\mathrm{Out}\,(G) \to \mathrm{Out}\,(G/H), \tag{1}$$

Then we say that H *separates torsion in* $\mathrm{Out}\,(G)$. In the case where $G = \mathbb{Z}^n$, the homomorphisms (1) include the maps $GL(n,\mathbb{Z}) \to GL(n,\mathbb{Z}/m\mathbb{Z})$. This motivates the terminology *congruences separate torsion in* G.

Separating torsion

We say that *congruence effectively separate torsion in* G if we can *algorithmically* find such a finite index characteristic subgroup $H \lhd_c G$.

Separating torsion

We say that *congruence effectively separate torsion in G* if we can *algorithmically* find such a finite index characteristic subgroup $H \lhd_c G$.

It is useful to think of $H \lhd_c G$ as being *deep enough* so that the kernel of $\mathrm{Out}(G) \to \mathrm{Out}(G/H)$ is torsion free.

The mixed Whitehead problem

The *mixed Whitehead problem* is a term coined by Bogopolski and Ventura and is easiest to formulate as a two quantifier problem:

Definition (The mixed Whitehead problem)

Let $(S_1, \ldots, S_k), (T_1, \ldots, T_k)$ be tuples of elements in G. *The mixed Whitehead problem* consists in deciding whether there exists $\sigma \in \mathrm{Aut}(G)$ and elements $g_1, \ldots, g_k \in G$ such that

$$\sigma(S_i) = T_i^{g_i}$$

for $i = 1, \ldots, k$. If such is the case we say the tuples of tuples $(S_1, \ldots, S_k), (T_1, \ldots, T_k)$ are *Whitehead equivalent*.

The mixed Whitehead problem

In the case of a single pair of tuples of elements, then the MWP asks if two tuples are in the same automorphic orbit. On the other extreme, if each tuple consists of a singleton, then this is the same as the classical Whitehead minimization for a tuple of elements.

But why?

Theorem (Dahmani-T)

Let $\mathcal{C}$ be a class of algorithmically tractable and effectively coherent groups (e.g. virtually polycyclic groups), satisfying the following properties:

- *$\mathcal{C}$ is closed for taking subgroups, and contains virtually cyclic groups*
- *all groups in $\mathcal{C}$ are residually finite,*
- *the isomorphism problem is explicitly solvable in $\mathcal{C}$,*
- *in $\mathcal{C}$, congruences effectively separate the torsion, and*
- *the mixed Whitehead problem is effectively solvable in $\mathcal{C}$.*

There is an algorithm which decides if two explicitly given torsion-free relatively hyperbolic groups $(G, \mathcal{P})$, $(H, \mathcal{Q})$ whose peripheral subgroups belongs to $\mathcal{C}$, are isomorphic as groups with unmarked peripheral structure.

But why?

Essentially what this says is that if the isomorphism problem is solvable in a class $\mathcal{C}$ of groups, then if congruences separate torsion, and if the MWP is solvable in this class, then we can lift this solution of the isomorphism problem for groups in $\mathcal{C}$ to the isomorphism problem for groups that are hyperbolic relative to groups in $\mathcal{C}$.

But why?

Essentially what this says is that if the isomorphism problem is solvable in a class $\mathcal{C}$ of groups, then if congruences separate torsion, and if the MWP is solvable in this class, then we can lift this solution of the isomorphism problem for groups in $\mathcal{C}$ to the isomorphism problem for groups that are hyperbolic relative to groups in $\mathcal{C}$.

This result therefore inserts itself in the sequence of results about relatively hyperbolic groups in which, if an algorithmic problem is solvable in the peripheral subgroups, then it is solvable in the ambient relatively hyperbolic group.

How do these come up?

Suppose we have two amalgams

$$\Gamma_1 = G_a *_{P_a} *P *_{P_b} G_b \text{ and } \Gamma_2 = G_a *_{P_a} *P *_{P_b} G_b$$

with the same vertex groups $(G_a, P_a), P, (G_b, P_b)$ but such that Γ_1 is constructed with attaching maps

$$i_1 : P_a \hookrightarrow P, j_1 : P_a \hookrightarrow P$$

and similarly Γ_2 is constructed with attaching maps i_2, j_2.

How do these come up?

These two graphs of groups with isomorphic vertex will be globally isomorphic if and only if there are automorphisms $\alpha \in \mathrm{Aut}\left(P\right)$ and β_a, β_b in $\mathrm{Aut}\left(G_a\right), \mathrm{Aut}\left(G_b\right)$, respectively such that

$$\alpha \circ i_1 \circ \beta_a \quad \sim_P \quad i_2 \tag{2}$$

$$\alpha \circ j_1 \circ \beta_b \quad \sim_P \quad j_2 \tag{3}$$

where $\sim_P$ denotes conjugacy in P.

How do these come up?

These two graphs of groups with isomorphic vertex will be globally isomorphic if and only if there are automorphisms $\alpha \in \mathrm{Aut}(P)$ and β_a, β_b in $\mathrm{Aut}(G_a), \mathrm{Aut}(G_b)$, respectively such that

$$\alpha \circ i_1 \circ \beta_a \sim_P i_2 \tag{2}$$
$$\alpha \circ j_1 \circ \beta_b \sim_P j_2 \tag{3}$$

where $\sim_P$ denotes conjugacy in P.

Note that α, β_a and β_b are all interdependent, if one changes then so must the other two.

How the MWP occurs

In the previous example suppose that $\mathrm{Out}\,(G_a)$ and $\mathrm{Out}\,(G_b)$ are trivial, and let X_a and X_b denote generating tuples for P_a and P_b in G_a and G_b.

How the MWP occurs

In the previous example suppose that $\mathrm{Out}\,(G_a)$ and $\mathrm{Out}\,(G_b)$ are trivial, and let X_a and X_b denote generating tuples for P_a and P_b in G_a and G_b. Denote by S_a, S_b the images $i_1(X_a), j_1(X_b)$ and by T_a, T_b the images $i_2(X_a), j_2(X_b)$. Then, because a homomorphism is specified by the image of a generating tuple, the formulas (2) (3) become

$$\begin{array}{ccc} \alpha(S_a) & \sim_P & T_a \\ \alpha(S_b) & \sim_p & T_b \end{array}$$

i.e. we get an instance of the MWP in P.

Why does do we want congruences to separate torsion?

In general to assume that the groups $(G_a, P_a), (G_b, P_b)$ have only trivial outer automorphisms is too much to ask. However if the pair, say, (G_a, P_a) is relatively hyperbolic and *rigid* then $\mathrm{Out}(G_a, P_a)$ is in fact finite.

Why does do we want congruences to separate torsion?

In general to assume that the groups $(G_a, P_a), (G_b, P_b)$ have only trivial outer automorphisms is too much to ask. However if the pair, say, (G_a, P_a) is relatively hyperbolic and *rigid* then $\mathrm{Out}\,(G_a, P_a)$ is in fact finite.

It turns out that if we take $H \lhd_c P_a$ that is sufficiently deep then by the hyperbolic Dehn filling Theorem of Osin or Manning-Groves the quotient $G_a/\langle\langle H\rangle\rangle$ will be a hyperbolic group.

Why does do we want congruences to separate torsion?

Furthermore, if H separates torsion in $\mathrm{Out}\,(P_a)$, then the image of the natural restriction of $\mathrm{Out}\,(G_a, P_a)$ to $\mathrm{Out}\,(P_a)$ is mapped injectively to $\mathrm{Out}\,(P_a/\langle\langle H\rangle\rangle)$ via the commutative diagram:

$$\begin{tikzcd}
\mathrm{Out}\,(G_a, P_a) \arrow[r] \arrow[d] & \mathrm{Out}\,(P_a) \arrow[d] \\
\mathrm{Out}\,(G_a/\langle\langle H\rangle\rangle, P_a/\langle\langle H\rangle\rangle) \arrow[r] & \mathrm{Out}\,(P_a/\langle\langle H\rangle\rangle)
\end{tikzcd}$$

Why does do we want congruences to separate torsion?

Furthermore, if H separates torsion in $\mathrm{Out}(P_a)$, then the image of the natural restriction of $\mathrm{Out}(G_a, P_a)$ to $\mathrm{Out}(P_a)$ is mapped injectively to $\mathrm{Out}(P_a/\langle\langle H\rangle\rangle)$ via the commutative diagram:

$$\begin{tikzcd}
\mathrm{Out}(G_a, P_a) \arrow[r] \arrow[d] & \mathrm{Out}(P_a) \arrow[d] \\
\mathrm{Out}(G_a/\langle\langle H\rangle\rangle, P_a/\langle\langle H\rangle\rangle) \arrow[r] & \mathrm{Out}(P_a/\langle\langle H\rangle\rangle)
\end{tikzcd}$$

Since we can effectively compute the bottom row of this diagram we are (after more work, and a deep result) able to reduce (2) (3) to finitely many instances of the MWP.

How do we separate torsion in $\mathrm{Out}(G)$?

Let G be a nilpotent group, then we denote by $\nu_i G$ the terms of the upper central series, i.e. $\nu_1 G = Z(G)$, $\nu_2 G$ is the preimage of the center of $G/\nu_1 G$, etc. We have a short exact series

$$1 \to \nu_1 G \to G \to G/\nu_1 G \to 1.$$

On one hand, since $\nu_1 G$ is characteristic, we have a projection

$$\mathrm{Out}(G) \xrightarrow{p} \mathrm{Out}(G/\nu_1 G)$$

the reason this only works for nilpotent groups...

On the other hand because $\nu_1 G$ is *central*, i.e. $\mathrm{Inn}(G)$ acts trivially, and has a *trivial inner automorphism group* we have a natural restriction map

$$\begin{array}{ccc} \mathrm{Aut}(G) & \longrightarrow & \mathrm{Aut}(\nu_1 G) \\ \downarrow & & \| \\ \mathrm{Out}(G) & \xrightarrow{r} & \mathrm{Out}(\nu_1 G) \end{array}$$

the reason this only works for nilpotent groups...

On the other hand because $\nu_1 G$ is *central*, i.e. $\mathrm{Inn}(G)$ acts trivially, and has a *trivial inner automorphism group* we have a natural restriction map

$$\begin{array}{ccc} \mathrm{Aut}(G) & \longrightarrow & \mathrm{Aut}(\nu_1 G) \\ \downarrow & & \| \\ \mathrm{Out}(G) & \xrightarrow{r} & \mathrm{Out}(\nu_1 G) \end{array}$$

This only works because $\nu_1 G$ is the center of G.

the reason this only works for nilpotent groups...

On the other hand because $\nu_1 G$ is *central*, i.e. $\mathrm{Inn}(G)$ acts trivially, and has a *trivial inner automorphism group* we have a natural restriction map

$$\begin{array}{ccc} \mathrm{Aut}(G) & \longrightarrow & \mathrm{Aut}(\nu_1 G) \\ \downarrow & & \parallel \\ \mathrm{Out}(G) & \xrightarrow{r} & \mathrm{Out}(\nu_1 G) \end{array}$$

This only works because $\nu_1 G$ is the center of G.

An element $[\beta] \in \mathrm{Out}\,(G)$ that vanishes in both the projection p and the restriction r is called *elusive*.

the reason this only works for nilpotent groups...

On the other hand because $\nu_1 G$ is *central*, i.e. $\mathrm{Inn}(G)$ acts trivially, and has a *trivial inner automorphism group* we have a natural restriction map

$$\begin{array}{ccc} \mathrm{Aut}(G) & \longrightarrow & \mathrm{Aut}(\nu_1 G) \\ \downarrow & & \| \\ \mathrm{Out}(G) & \xrightarrow{r} & \mathrm{Out}(\nu_1 G) \end{array}$$

This only works because $\nu_1 G$ is the center of G.

An element $[\beta] \in \mathrm{Out}(G)$ that vanishes in both the projection p and the restriction r is called *elusive*. Ultimately our algorithm will be by induction on the upper central series length, but this map r can only be guaranteed to exist if G is nilpotent.

Making non-elusive elements survive

Suppose we are given subgroups K_0 and N_0 that separated torsion in $\mathrm{Out}(N/\nu_1 N), \mathrm{Out}(\nu_1 N)$ respectively, then we can find a finite index characteristic subgroup $P_0 \lhd_c G$ such that $P_0 \cap \nu_1 N \leqslant N_0$ and $P_0 \leqslant \nu_1 G K_0$ so that non-elusive elements survive in the quotient

$$\mathrm{Out}\,(G) \to \mathrm{Out}\,(G/P_0).$$

Making non-elusive elements survive

Suppose we are given subgroups K_0 and N_0 that separated torsion in $\mathrm{Out}(N/\nu_1 N), \mathrm{Out}(\nu_1 N)$ respectively, then we can find a finite index characteristic subgroup $P_0 \lhd_c G$ such that $P_0 \cap \nu_1 N \leqslant N_0$ and $P_0 \leqslant \nu_1 G K_0$ so that non-elusive elements survive in the quotient

$$\mathrm{Out}\,(G) \to \mathrm{Out}\,(G/P_0).$$

The argument isn't completely trivial, but it's not difficult. The main trick is that nilpotent groups are subgroup separable.

Characterizing elusive elements

Lemma

Each $\xi \in \nu_2 G$ *induces a homomorphism* $z_\xi : G \to \nu_1 G$ *given by the mapping*

$$x \mapsto [x, \xi].$$

Moreover the mapping

$$\Phi : \nu_2 G \to \mathrm{Hom}(G, \nu_1 G)$$

is in fact a homomorphism where $\mathrm{Hom}(G, \nu_1 G)$ *is viewed as an abelian group with equipped with the standard* $\mathbb{Z}$*-module addition.*

Characterizing elusive elements

Proof.

For all $x \in N, \xi \in \nu_2 N$ we have $[x,\xi] = z_\xi(x) \in \nu_1 N$. Let $x, y \in N$ then with the commutator convention $[x,y] = x^{-1}y^{-1}xy$ we can observe that on one hand $(xy)\xi = \xi(x,y)[(xy),\xi] = \xi(xy)z_\xi(xy)$ and on the other hand (recall that $[z,\xi]$ is always central):

$$xy\xi = x\xi y[y,\xi] = \xi x[x,\xi]y[y,\xi] = \xi(xy)[x,\xi][y,\xi] = \xi(xy)z_\xi(x)z_\xi(y)$$

so the map $x \mapsto z_\xi(x)$ is a homomorphism.
Let now $\xi, \zeta \in \nu_2 N$ and let $x \in N$ one hand setting $[x,\xi\zeta] = z_{\xi\zeta}(x)$ we have $x(\xi\zeta) = (\xi\zeta)xz_{\xi\zeta}(x)$ and on the other hand we have:

$$x\xi\zeta = \xi x z_\xi(x)\zeta = \xi x\zeta z_\xi(x) = (\xi\zeta)xz_\zeta(x)z_\xi(x)$$

which gives the formula: $z_{\xi\zeta}(x) = z_\xi(x) + z_\zeta(x)$ so the map Φ is a homomorphism. $\square$

Characterizing elusive elements

On the other hand we have:

Lemma

Let

$$\mathrm{Hom}^*(N, \nu_1 N) = \{f \in Hom(N, \nu_1 N) \mid \nu_1 N \leqslant \ker(f)\}.$$

For each $f \in \mathrm{Hom}^*(N, \nu_1 N)$ *the map* $x \mapsto xf(x)$ *is an automorphism* $\Psi(f) \in \mathrm{Aut}(N)$. *Moreover this map* $\Psi : \mathrm{Hom}^*(N, \nu_1 N) \to \mathrm{Aut}(N)$ *is a homomorphism.*

Characterizing elusive elements

On the other hand we have:

Lemma

Let

$$\mathrm{Hom}^*(N, \nu_1 N) = \{f \in Hom(N, \nu_1 N) \mid \nu_1 N \leqslant \ker(f)\}.$$

For each $f \in \mathrm{Hom}^*(N, \nu_1 N)$ *the map* $x \mapsto xf(x)$ *is an automorphism* $\Psi(f) \in \mathrm{Aut}(N)$. *Moreover this map* $\Psi : \mathrm{Hom}^*(N, \nu_1 N) \to \mathrm{Aut}(N)$ *is a homomorphism.*

Furthermore the composition

$$\Psi \circ \Phi : \nu_2 G \to \mathrm{Aut}(G)$$

sends ξ to γ_ξ, conjugation by ξ.

Characterizing elusive elements

If an automorphism β has a finite order image $[\beta] \in \mathrm{Out}(G)$ then this means that $\beta^n = \gamma_\xi$ for some $\xi \in G$. Morally β can be seen as the n^{th} root of ξ. We show

Characterizing elusive elements

If an automorphism β has a finite order image $[\beta] \in \mathrm{Out}\,(G)$ then this means that $\beta^n = \gamma_\xi$ for some $\xi \in G$. Morally β can be seen as the n^{th} root of ξ. We show

Proposition

The set of elusive elements of $\mathrm{Out}(G)$ coincides exactly with the set

$$\left\{[\beta] \in \mathrm{Out}(N) \mid (\exists \beta \in [\beta])\ \beta \in \Psi(\hat{S} \setminus S)\right\}$$

where $S = \Phi(\nu_2 G)$ and $S \leqslant \hat{S}$ is the set consisting of $f \in \mathrm{Hom}^*(G, \nu_1 G)$ such that

$$d \cdot f = \underbrace{f + \cdots + f}_{d\ times} \in S = \Phi(\nu_2 G)$$

for some $d \in \mathbb{Z}_{\geqslant 0}$, i.e. the isolator of $\Phi(\nu_2 G)$ in $\mathrm{Hom}^*(G, \nu_1 G)$.

An existential problem

So far all the objects and maps mentioned are computable. That being said we still do not know if there exists a deep enough finite index $H \lhd_c G$ such that elusive elements do not vanish in $\mathrm{Out}\,(G/H)$.

An existential problem

So far all the objects and maps mentioned are computable. That being said we still do not know if there exists a deep enough finite index $H \lhd_c G$ such that elusive elements do not vanish in $\mathrm{Out}\,(G/H)$. We therefore asked for help. . .

An existential problem

So far all the objects and maps mentioned are computable. That being said we still do not know if there exists a deep enough finite index $H \triangleleft_c G$ such that elusive elements do not vanish in $\mathrm{Out}\,(G/H)$. We therefore asked for help...

Proposition (Segal)

Let P be a virtually polycyclic group and denote by $P_n \triangleleft_c P$ be a sequence of finite index subgroups which eventually lie inside any fixed finite index subgroup. For every finite order $[\alpha] \in \mathrm{Out}\,(P)$ there exists some j such that for every $k \geqslant j$ the image $\overline{[\alpha]}_k \in \mathrm{Out}\,(P/P_k)$ is non-trivial.

An existential problem

So far all the objects and maps mentioned are computable. That being said we still do not know if there exists a deep enough finite index $H \triangleleft_c G$ such that elusive elements do not vanish in $\mathrm{Out}(G/H)$. We therefore asked for help...

Proposition (Segal)

Let P be a virtually polycyclic group and denote by $P_n \triangleleft_c P$ be a sequence of finite index subgroups which eventually lie inside any fixed finite index subgroup. For every finite order $[\alpha] \in \mathrm{Out}(P)$ there exists some j such that for every $k \geqslant j$ the image $\overline{[\alpha]}_k \in \mathrm{Out}(P/P_k)$ is non-trivial.

The proof involves passing to the profinite completion and applying a deep result about closures of centralizers in profinite completions of virtually polycyclic groups due to Ribes-Segal-Zalesski.

Separating torsion in $\mathrm{Out}(G)$

So by the previous result, we know that by enumerating characteristic finite quotients, we will eventually get a deep enough subgroup, and by the computable algebraic characterization we know what to look for, and therefore when to stop.

Separating torsion in $\mathrm{Out}(G)$

So by the previous result, we know that by enumerating characteristic finite quotients, we will eventually get a deep enough subgroup, and by the computable algebraic characterization we know what to look for, and therefore when to stop. $\square$

The mixed Whitehead problem, orbits of arithmetic groups

Our solution to the mixed Whitehead relies on the work of Grunewald and Segal on the decidability of orbit problems. Here is one of their fundamental results:

Theorem (Grunewald-Segal)

There exists and algorithm which takes as input:

- *an explicitly given $\mathbb{Q}$-defined algebraic group $\mathscr{G}$,*

The algorithm decides whether there is some $\gamma \in \Gamma$ such that $\rho(\gamma) \cdot a = b$ and if so produces such a matrix γ.

The mixed Whitehead problem, orbits of arithmetic groups

Our solution to the mixed Whitehead relies on the work of Grunewald and Segal on the decidability of orbit problems. Here is one of their fundamental results:

Theorem (Grunewald-Segal)

There exists and algorithm which takes as input:

- *an explicitly given $\mathbb{Q}$-defined algebraic group $\mathscr{G}$,*
- *ρ, an explicitly given rational action on some subset $W \subset \mathbb{C}^n$,*

The algorithm decides whether there is some $\gamma \in \Gamma$ such that $\rho(\gamma) \cdot a = b$ and if so produces such a matrix γ.

The mixed Whitehead problem, orbits of arithmetic groups

Our solution to the mixed Whitehead relies on the work of Grunewald and Segal on the decidability of orbit problems. Here is one of their fundamental results:

Theorem (Grunewald-Segal)

There exists and algorithm which takes as input:

- *an explicitly given $\mathbb{Q}$-defined algebraic group $\mathscr{G}$,*
- *ρ, an explicitly given rational action on some subset $W \subset \mathbb{C}^n$,*
- *Γ, an explicitly given arithmetic subgroup of $\mathscr{G}$, and*

The algorithm decides whether there is some $\gamma \in \Gamma$ such that $\rho(\gamma) \cdot a = b$ and if so produces such a matrix γ.

The mixed Whitehead problem, orbits of arithmetic groups

Our solution to the mixed Whitehead relies on the work of Grunewald and Segal on the decidability of orbit problems. Here is one of their fundamental results:

Theorem (Grunewald-Segal)

There exists and algorithm which takes as input:

- *an explicitly given* $\mathbb{Q}$*-defined algebraic group* $\mathscr{G}$*,*
- ρ*, an explicitly given rational action on some subset* $W \subset \mathbb{C}^n$*,*
- Γ*, an explicitly given arithmetic subgroup of* $\mathscr{G}$*, and*
- *two points* $a, b \in W \cap \mathbb{Q}^n$

The algorithm decides whether there is some $\gamma \in \Gamma$ *such that* $\rho(\gamma) \cdot a = b$ *and if so produces such a matrix* γ*.*

The mixed Whitehead problem, representations of nilpotent groups

Theorem (Grunewald-Segal)

Given a finite presentation $\langle X \mid R \rangle$ of a $\mathfrak{T}$-group G we can effectively find a suitable n and an embedding

$$\Theta_G : G \hookrightarrow \mathrm{Tr}_1(n, \mathbb{Z})$$

such that the natural map

$$N_{GL(n,\mathbb{Z})}(\Theta_G(G)) \to \mathrm{Aut}\,(G)$$

is surjective.

The mixed Whitehead problem, representations of nilpotent groups

Theorem (Grunewald-Segal)

Given a finite presentation $\langle X \mid R \rangle$ of a $\mathfrak{T}$-group G we can effectively find a suitable n and an embedding

$$\Theta_G : G \hookrightarrow \mathrm{Tr}_1(n, \mathbb{Z})$$

such that the natural map

$$N_{GL(n,\mathbb{Z})}(\Theta_G(G)) \to \mathrm{Aut}\,(G)$$

is surjective.

Furthermore the normalizer $N_{GL(n,\mathbb{Z})}(\Theta_G(G))$ can be explicitly given as an arithmetic group.

The mixed Whitehead problem, representations of nilpotent groups

Theorem (Grunewald-Segal)

Given a finite presentation $\langle X \mid R \rangle$ of a $\mathfrak{T}$-group G we can effectively find a suitable n and an embedding

$$\Theta_G : G \hookrightarrow \mathrm{Tr}_1(n, \mathbb{Z})$$

such that the natural map

$$N_{GL(n,\mathbb{Z})}(\Theta_G(G)) \to \mathrm{Aut}\,(G)$$

is surjective.

Furthermore the normalizer $N_{GL(n,\mathbb{Z})}(\Theta_G(G))$ can be explicitly given as an arithmetic group. So already we can use the orbit algorithm to decide if $g, h \in G$ are in the same $\mathrm{Aut}\,(G)$ orbit.

How to solve the mixed Whitehead problem

Recall the mixed Whitehead problem:

Definition (The mixed Whitehead problem)

Let $(S_1, \ldots, S_k), (T_1, \ldots, T_k)$ be tuples of elements in G. *The mixed Whitehead problem* consists in deciding whether there exists $\sigma \in \mathrm{Aut}(G)$ and elements $g_1, \ldots, g_k \in G$ such that

$$\sigma(S_i) = T_i^{g_i}$$

for $i = 1, \ldots, k$. If such is the case we say the tuples of tuples $(S_1, \ldots, S_k), (T_1, \ldots, T_k)$ are *Whitehead equivalent*.

Reduction to a one quantifier problem

Let G be some group and let $\Gamma = \mathrm{Aut}(G)$. There is a well defined right action of the right semidirect product $\Gamma \ltimes G^r$ on the set of r-tuples of tuples given by

$$(S_1, \ldots, S_r) \cdot \big(\sigma; (g_1, \ldots, g_r)\big) = (\sigma^{-1}(S_1)^{g_1}, \ldots, \sigma^{-1}(S_r)^{g_r}). \qquad (4)$$

It immediately follows that

Reduction to a one quantifier problem

Let G be some group and let $\Gamma = \mathrm{Aut}\,(G)$. There is a well defined right action of the right semidirect product $\Gamma \ltimes G^r$ on the set of r-tuples of tuples given by

$$(S_1, \ldots, S_r) \cdot (\sigma; (g_1, \ldots, g_r)) = (\sigma^{-1}(S_1)^{g_1}, \ldots, \sigma^{-1}(S_r)^{g_r}). \qquad (4)$$

It immediately follows that

Proposition

Let $(S_1, \ldots, S_k), (T_1, \ldots, T_k)$ be tuples of elements in G. They are Whitehead equivalent if and only if they lie in the same orbit under the $\Gamma \ltimes G^r$-action given in (4).

If we identify $G = \Theta_G(G) \leqslant \mathrm{Tr}_1(n, \mathbb{Z})$, then there is a natural surjection $N_{GL(n,\mathbb{Z})}(G) \ltimes G^k \twoheadrightarrow \Gamma \ltimes G^k$ and the map

$$(r; (h_1, \ldots, h_k)) \mapsto \mathrm{diag}(r, rh_k, \ldots, rh_k)$$

which sends elements of $N_{GL(n,\mathbb{Z})}(G) \ltimes G^k$ to block matrices with zero everywhere except the $n \times n$ matrices $r, rh_1, \ldots, rh_k$ along the main diagonal gives a linear representation.

If we identify $G = \Theta_G(G) \leqslant \mathrm{Tr}_1(n, \mathbb{Z})$, then there is a natural surjection $N_{GL(n,\mathbb{Z})}(G) \ltimes G^k \twoheadrightarrow \Gamma \ltimes G^k$ and the map

$$(r; (h_1, \ldots, h_k)) \mapsto \mathrm{diag}(r, rh_k, \ldots, rh_k)$$

which sends elements of $N_{GL(n,\mathbb{Z})}(G) \ltimes G^k$ to block matrices with zero everywhere except the $n \times n$ matrices $r, rh_1, \ldots, rh_k$ along the main diagonal gives a linear representation.

Furthermore, since G and $N_{GL(n,\mathbb{Z})}(G)$ are explicitly given arithmetic groups, the this gives also gives $N_{GL(n,\mathbb{Z})}(G) \ltimes G^k$ explicitly as an arithmetic group, which acts by conjugation on G^K (which is supposed to encode our set of tuples of tuples), represented as a group of block matrices.

If we identify $G = \Theta_G(G) \leqslant \mathrm{Tr}_1(n, \mathbb{Z})$, then there is a natural surjection $N_{GL(n,\mathbb{Z})}(G) \ltimes G^k \twoheadrightarrow \Gamma \ltimes G^k$ and the map

$$(r; (h_1, \ldots, h_k)) \mapsto \mathrm{diag}(r, rh_k, \ldots, rh_k)$$

which sends elements of $N_{GL(n,\mathbb{Z})}(G) \ltimes G^k$ to block matrices with zero everywhere except the $n \times n$ matrices $r, rh_1, \ldots, rh_k$ along the main diagonal gives a linear representation.

Furthermore, since G and $N_{GL(n,\mathbb{Z})}(G)$ are explicitly given arithmetic groups, the this gives also gives $N_{GL(n,\mathbb{Z})}(G) \ltimes G^k$ explicitly as an arithmetic group, which acts by conjugation on G^K (which is supposed to encode our set of tuples of tuples), represented as a group of block matrices.

We can therefore use the Grunewald-Segal orbit decidability algorithm to solve the mixed Whitehead problem.

If we identify $G = \Theta_G(G) \leqslant \mathrm{Tr}_1(n, \mathbb{Z})$, then there is a natural surjection $N_{GL(n,\mathbb{Z})}(G) \ltimes G^k \twoheadrightarrow \Gamma \ltimes G^k$ and the map

$$(r; (h_1, \ldots, h_k)) \mapsto \mathrm{diag}(r, rh_k, \ldots, rh_k)$$

which sends elements of $N_{GL(n,\mathbb{Z})}(G) \ltimes G^k$ to block matrices with zero everywhere except the $n \times n$ matrices $r, rh_1, \ldots, rh_k$ along the main diagonal gives a linear representation.

Furthermore, since G and $N_{GL(n,\mathbb{Z})}(G)$ are explicitly given arithmetic groups, the this gives also gives $N_{GL(n,\mathbb{Z})}(G) \ltimes G^k$ explicitly as an arithmetic group, which acts by conjugation on G^K (which is supposed to encode our set of tuples of tuples), represented as a group of block matrices.

We can therefore use the Grunewald-Segal orbit decidability algorithm to solve the mixed Whitehead problem. By the way torsion in G is unproblematic.

Where do we go from here?

The machinery just presented is what is needed to solve the isomorphism problem for nilpotent groups.

Where do we go from here?

The machinery just presented is what is needed to solve the isomorphism problem for nilpotent groups.

Segal generalized this to prove the isomorphism problem for polycyclic-by-finite groups.

Where do we go from here?

The machinery just presented is what is needed to solve the isomorphism problem for nilpotent groups.

Segal generalized this to prove the isomorphism problem for polycyclic-by-finite groups. At the time when I first looked at his paper *Decidable properties of polycyclic groups*, I could not for remember the definition nor the significance of an arithmetic group, but last night upon rereading I think that the same approach will work for polycyclic groups where Theorem G (which is strangely formulated) is used in place of the orbit decidability algorithm, but this needs to be verified carefully.

What about separating torsion in $\operatorname{Out}(G)$?

Grunewald and Baues proved in 2005 that if G is polycyclic-by-finite then $\operatorname{Out}(G)$ is in fact arithmetic. Even though $\operatorname{Aut}(G)$ may not even contain an arithmetic subgroup of finite index.

What about separating torsion in $\mathrm{Out}(G)$?

Grunewald and Baues proved in 2005 that if G is polycyclic-by-finite then $\mathrm{Out}(G)$ is in fact arithmetic. Even though $\mathrm{Aut}(G)$ may not even contain an arithmetic subgroup of finite index.

It is not clear, however, if their representation of $\mathrm{Out}(G)$ as an arithmetic group can be made explicit, or if given a generating set $\{\gamma_1, \ldots, \gamma_n\}$ for $\mathrm{Aut}(G)$ whether the images (as matrices) $[\gamma_i] \in \mathrm{Out}(G)$ can be computed.

What about separating torsion in $\mathrm{Out}(G)$?

Grunewald and Baues proved in 2005 that if G is polycyclic-by-finite then $\mathrm{Out}(G)$ is in fact arithmetic. Even though $\mathrm{Aut}(G)$ may not even contain an arithmetic subgroup of finite index.

It is not clear, however, if their representation of $\mathrm{Out}(G)$ as an arithmetic group can be made explicit, or if given a generating set $\{\gamma_1, \ldots, \gamma_n\}$ for $\mathrm{Aut}(G)$ whether the images (as matrices) $[\gamma_i] \in \mathrm{Out}(G)$ can be computed.

If this is the case however, then modulo the production of a complete list of conjugacy representatives of finite order elements of $\mathrm{Out}(G)$ (we can do that, right? effective Borel-Harish-Chandra or something), we can enumerate finite quotients and we'll know when to stop.

What about separating torsion in $\mathrm{Out}(G)$?

Grunewald and Baues proved in 2005 that if G is polycyclic-by-finite then $\mathrm{Out}(G)$ is in fact arithmetic. Even though $\mathrm{Aut}(G)$ may not even contain an arithmetic subgroup of finite index.

It is not clear, however, if their representation of $\mathrm{Out}(G)$ as an arithmetic group can be made explicit, or if given a generating set $\{\gamma_1, \ldots, \gamma_n\}$ for $\mathrm{Aut}(G)$ whether the images (as matrices) $[\gamma_i] \in \mathrm{Out}(G)$ can be computed.

If this is the case however, then modulo the production of a complete list of conjugacy representatives of finite order elements of $\mathrm{Out}(G)$ (we can do that, right? effective Borel-Harish-Chandra or something), we can enumerate finite quotients and we'll know when to stop. I.e. we will have found a deep enough finite index characteristic subgroup, where no finite order outer automorphism vanishes.

The actual question I wanted to investigate

On one hand, Fromanek and Remeslennikov showed that polycyclic-by-finite groups are conjugacy separable, but that Segal showed that it is possible to construct arbitrarily large (but finite) sets of pairwise non-isomorphic polycyclic-by-finite with isomorphic finite quotients.

The actual question I wanted to investigate

On one hand, Fromanek and Remeslennikov showed that polycyclic-by-finite groups are conjugacy separable, but that Segal showed that it is possible to construct arbitrarily large (but finite) sets of pairwise non-isomorphic polycyclic-by-finite with isomorphic finite quotients.

What remains unknown (to me) however is if $g, h \in G$ and for every finite quotient of G there is an automorphism which brings $\bar{g}$ to $\bar{h}$ if this can lift to an automorphism of G with the desired property.

The actual question I wanted to investigate

On one hand, Fromanek and Remeslennikov showed that polycyclic-by-finite groups are conjugacy separable, but that Segal showed that it is possible to construct arbitrarily large (but finite) sets of pairwise non-isomorphic polycyclic-by-finite with isomorphic finite quotients.

What remains unknown (to me) however is if $g, h \in G$ and for every finite quotient of G there is an automorphism which brings $\bar{g}$ to $\bar{h}$ if this can lift to an automorphism of G with the desired property.

This happens to be the case for hyperbolic Dehn fillings of *rigid* relatively hyperbolic groups according to Dahmani and Guirardel. Their argument relies on Gromov-Hausdorff limits, instead of inverse limits.

The actual question I wanted to investigate

On one hand, Fromanek and Remeslennikov showed that polycyclic-by-finite groups are conjugacy separable, but that Segal showed that it is possible to construct arbitrarily large (but finite) sets of pairwise non-isomorphic polycyclic-by-finite with isomorphic finite quotients.

What remains unknown (to me) however is if $g, h \in G$ and for every finite quotient of G there is an automorphism which brings $\bar{g}$ to $\bar{h}$ if this can lift to an automorphism of G with the desired property.

This happens to be the case for hyperbolic Dehn fillings of *rigid* relatively hyperbolic groups according to Dahmani and Guirardel. Their argument relies on Gromov-Hausdorff limits, instead of inverse limits. Can Gromov-Hausdorff limit of finite quotients of polycyclic-by-finite groups be useful?

Am I just being silly?

Consider $\mathbb{Z}$. It's profinite completion has a huge automorphism group, but if we look at asymptotic cones (okay, ultralimits of scaled based metric spaces) of $\mathbb{Z}/n\mathbb{Z}$ the ultralimits are either $\mathbb{Z}$ or $\mathbb{R}$ and our limiting automorphism will act by an isometry fixing a point, i.e. will have order 2.

Thank you!

Thank you!