$\mathbb{Z}^n$-free groups are CAT(0)

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Lyndon Length Function

Let G be a group and let Λ be a totally ordered abelian group.

Definition

A Lyndon length function is a mapping $\ell : G \to \Lambda$ such that

(L1) $\forall\, g \in G : \ \ell(g) \geqslant 0$ and $\ell(1) = 0$;

(L2) $\forall\, g \in G : \ \ell(g) = \ell(g^{-1})$;

(L3) $\forall\, g, f, h \in G : \ c(g, f) > c(g, h) \to c(g, h) = c(f, h)$,
where $c(g, f) = \frac{1}{2}(\ell(g) + \ell(f) - \ell(g^{-1}f))$.

Free Regular Lyndon Length Function

The length function $\ell : G \to \Lambda$ is *free* if also

(L4) $\forall\, g, f \in G :\ c(g, f) \in \Lambda.$

(L5) $\forall\, g \in G :\ g \neq 1 \to \ell(g^2) > \ell(g).$

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The length function $\ell : G \to \Lambda$ is *regular* if also

(L6) $\forall\ g, f \in G,\ \exists\ u, g_1, f_1 \in G:$
$g = ug_1,\ \ell(g) = \ell(u) + \ell(g_1);\ f = uf_1$
$\ell(f) = \ell(u) + \ell(f_1);\ \ell(u) = c(g, f).$

Λ-free group

Definition

By a Λ-*free* group we mean a finitely generated group G equipped with a free Lyndon Λ-length function. We call G a *regular* Λ-*free* group if the length function is free and regular.

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Definition

A Λ-*tree* is a geodesic Λ-metric space such that:

1. If two segments in X intersect in a single point, which is an endpoint of both, then their union is a segment,
2. The intersection of two segments with a common endpoint is also a segment.

Λ-*free* groups can be thought of as groups acting freely on Λ-trees, and regular Λ-*free* groups are precisely those acting with a unique orbit of branch points.

Group actions on Λ-trees

Some history

- introduced by Morgan and Shalen in 1984.
- studied by Alperin and Bass. In 1991 Bass proved a version of a combination theorem for finitely generated groups acting freely on $(\Lambda \oplus \mathbb{Z})$-trees. For instance, by Bass' Theorem, the group $G = \langle x_1, x_1^2 x_2 \rangle *_{\langle x_1^2 x_2 = (x_2 x_3^2)^{-1} \rangle} \langle x_2 x_3^2, x_3 \rangle$ is $\mathbb{Z}^2$-free. Note that $G = \langle x_1, x_2, x_3 \mid x_1^2 x_2^2 x_3^2 = 1 \rangle$.
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Major progress

- Work by Chiswell.
- Work by Kharlampovich, Miasnikov, Remeslennikov and Serbin.

Examples of Λ-free groups

- $\mathbb{Z}$-free groups are f.g. free groups (Bass-Serre theory).
- $\mathbb{R}$-free groups are free products of free abelian groups and surface groups (Rips' theory).
- Let F be a free group and let $u, v \in F$ be such that $|u| = |v|$ and u is not conjugate to v^{-1}. Then $G = \langle F, t \mid tut^{-1} = v \rangle$ is a regular $\mathbb{Z}^2$-free group (follows from a recent result by Miasnikov, Remeslennikov and Serbin).
- F.g. fully residually free (or limit) groups act freely on $\mathbb{Z}^n$-trees with the lexicographic order (Kharlampovich and Miasnikov, 1998)

Basic properties of $\mathbb{Z}^n$-free groups

- Torsion-free,
- Commutative transitive:
  $[a, b] = 1, [b, c] = 1 \Rightarrow [a, c] = 1 \; \forall a, b, c \in G$,

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Definition

A group G is *residually free* if for any $1 \neq g \in G$ there is $\phi \colon G \to F$ so that $\phi(g) \neq 1$. G is *fully residually free* if for any finite set of elements $g_1, \ldots, g_m \in G$ there is $\phi \colon G \to F$ so that the images $\phi(g_1), \ldots, \phi(g_m)$ are all distinct.

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Remark. By a theorem of B.Baumslag, a *residually free* group G is fully residually free if and only if G is commutative transitive.

Non-residually free $\mathbb{Z}^n$-free groups

Theorem (Kharlampovich and Miasnikov, 1998)

F.g. fully residually free (or limit) groups act freely on $\mathbb{Z}^n$-trees with the lexicographic order.

Example. A $\mathbb{Z}^2$-free group that is not residually free:

$G = \langle x_1, x_2, x_3 \mid x_1^2 x_2^2 x_3^2 = 1 \rangle$

In a free group, $x_1^2 x_2^2$ is not a proper square unless $[x_1, x_2] = 1$ hence, G is not residually free.

We have seen that G is $\mathbb{Z}^2$-free.

$\mathbb{Z}^n$-free groups are relatively hyperbolic

Theorem (Guirardel, 2004)

A f.g. freely indecomposable $\mathbb{R}^n$-free group G is the fundamental group of a finite graph of groups with cyclic edge groups, where each vertex group is a f.g. $\mathbb{R}^{n-1}$-free subgroup of G.

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From this and the combination theorem for relatively hyperbolic groups (F. Dahmani; E. Alibegovič):

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Theorem (Nikolaev, 2010)

If G is a $\mathbb{Z}^n$-free group then the membership problem for finitely generated subgroups of G is decidable.

CAT(0) spaces

Let X be a geodesic metric space, and let $x_1, x_2, x_3 \in X$ be three points in X.
Consider a geodesic triangle $x_1x_2x_3$ in X and a *comparison triangle* $x_1'x_2'x_3'$ in the Euclidean plane E^2: $d_X(x_i, x_j) = d_E(x_i', x_j')$, for all i, j.

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Choose two arbitrary points $y_1 \in [x_3, x_1]$ and $y_2 \in [x_3, x_2]$, and let $y_i' \in [x_3', x_i']$ be such that $d_X(x_3, y_i) = d_E(x_3', y_i')$ for $i = 1, 2$.

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Definition

We say that X *satisfies the CAT(0) inequality* if $d_X(y_1, y_2) \leq d_E(y_1', y_2')$, for any choice of x_1, x_2, x_3, y_1 and y_2 in X. The space X is *locally CAT(0)* if every point $x \in X$ has a neighborhood that satisfies the CAT(0) inequality. By a CAT(0) space we mean a simply connected locally CAT(0) space.

Examples of CAT(0) spaces:

- Simplicial trees, where each edge is assigned a finite length.
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Examples of locally CAT(0) spaces:

- Finite graphs, where each edge is assigned a finite length.
- Tori T^n, for all integer n.

CAT(0) groups

Let G be a discrete group acting on a topological space X.

Definition

We say that an action of G on X is *properly discontinuous* if every point $x \in X$ has a neighbourhood U_x such that there are only finitely many elements $g \in G$ with $g.U_x \cap U_x \neq \emptyset$.
We say that the action of G on X is *cocompact* if the quotient $G \backslash X$ is compact.

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Definition

A group G is called a *CAT(0) group* if it acts properly discontinuously and cocompactly on a CAT(0) space.

Examples of CAT(0) groups

- Finitely generated free groups.
- Finitely generated free abelian groups.
- Free products of CAT(0) groups.
- (Alibegović and Bestvina, 2006) Limit groups.

Main Results

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Theorem (Alibegović and Bestvina, 2006)

A limit group G acts properly discontinuously and cocompactly on a CAT(0) space with isolated flats.

Motivation: Are (word) hyperbolic groups CAT(0)?

Main Theorem

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If G is a regular $\mathbb{Z}^n$-free group then G is the fundamental group of a compact connected locally CAT(0) geometrically coherent space.

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Definition (Alibegović, Bestvina; Wise)

Let X be a connected locally CAT(0) space, and let C be a connected subspace of X. C is a *core* of X if C is compact, locally CAT(0), and the inclusion $C \hookrightarrow X$ induces a π_1-isomorphism.
Let Y be a connected locally CAT(0) space. Y is called *geometrically coherent* if every covering space $X \to Y$ with X connected and $\pi_1(X)$ finitely generated has the following property. For every compact subset $K \subset X$ there is a core C of X containing K.

Embedding

Theorem (Kharlampovich, Miasnikov, Remeslennikov and Serbin 2010)

Every f.g. $\mathbb{Z}^n$-free group embeds by a length-preserving monomorphism into a f.g. regular $\mathbb{Z}^m$-free group, for some m.

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Definition

A group G is called *coherent* if every finitely generated subgroup of G is finitely presented.

Corollary

If G is a regular $\mathbb{Z}^n$-free group then G is coherent.

Remark. The coherence of a regular $\mathbb{Z}^n$-free group G follows from the structure theorem [KMRS].

Gluing locally CAT(0) spaces

Lemma (Bridson and Haefliger, 1999)

Let X and A be locally CAT(0) metric spaces. If A is compact and $\varphi, \phi \colon A \to X$ are local isometries, then the quotient of $X \coprod (A \times [0,1])$ by the equivalence relation generated by

$$[(a,0) \sim \varphi(a); (a,1) \sim \phi(a)],$$

for all $a \in A$, is locally CAT(0).

Using this lemma, one can show that regular $\mathbb{Z}^n$-free groups are fundamental groups of compact connected locally CAT(0) spaces and therefore, are CAT(0) groups.

Structure Theorem for regular $\mathbb{Z}^n$-free groups

Theorem (Kharlampovich, Miasnikov, Remeslennikov and Serbin 2010)

Let G be a f.g. group with a regular free Lyndon length function $\ell : G \to \mathbb{Z}^n$. Then G can be represented as a union of a finite series of groups $F_m = G_1 < G_2 < \cdots < G_r = G$, so that

$$G_{i+1} = \langle G_i, s_i \mid s_i^{-1} \ C_i \ s_i = \phi_i(C_i) \rangle,$$

where the maximal abelian subgroup C_i of G_i and the isomorphism ϕ_i are carefully chosen.
In particular, $\ell(\phi_i(w)) = \ell(w) \ \forall w \in C_i$.

Key Technical Lemma

Lemma (B., Kharlampovich, 2013)

Let G be a regular $\mathbb{Z}^n$-free group. One can assign positive integer weights to the generators $g_i \in Y$ of G so that in every HNN-extension $t^{-1}Ct = \phi(C)$ in the statement of Theorem [KMRS], for every element $g \in C$, $wm(g) = wm(\phi(g))$.

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Theorem (Gurevich and Kokorin, 1963)

Any two nontrivial ordered abelian groups satisfy the same existential sentences in the language $L = \{0, +, -, <\}$.

$$\sum_{i=1}^{m_u} \ell(g_i) = \sum_{i=1}^{m_v} \ell(f_i)$$

$$\ell(g_i) > 0, \ \forall g_i \in Y.$$

The CAT(0) space: one step in the construction

Theorem

Let U be a geometrically coherent locally CAT(0) space, let T be a k-torus for some integer $k \geq 1$, and let $\varphi\colon T \to U$ and $\phi\colon T \to U$ be local isometries. Furthermore, assume that both images $\varphi(T)$ and $\phi(T)$ are locally convex, maximal and separated in U. We also assume that if $\varphi(T) \neq \phi(T)$ then the $\pi_1(U)$-orbits of $\varphi(T)$ and $\phi(T)$ are disjoint.
Let Y be the quotient of

$$U \coprod T \times [0,1]$$

by the equivalence relation generated by

$$[(a,0) \sim \varphi(a),\ (a,1) \sim \phi(a), \forall a \in T].$$

Then Y is locally CAT(0) and geometrically coherent.

The CAT(0) space: the base step

U_1 is the wedge of circles corresponding to the free group $F = G_1$. Every maximal abelian subgroup C of G_1 is cyclic: $C = \langle c \rangle$. By the Key Technical Lemma, $wm(c) = wm(\phi(c))$.

We rescale the metric on S^1 and let $\tau\colon S^1 \to \alpha$ and $\sigma\colon S^1 \to \beta$ be local isometries.
U_2 is the quotient of $U_1 \coprod S^1 \times [0,1]$ by the equivalence relation generated by $[(a,0) \sim \tau(a),\ (a,1) \sim \sigma(a), \forall a \in S^1]$.

If $c \neq \phi(c)$ then c^{-1} and $\phi(c)$ are not conjugate in G_1.

By the previous Theorem, U_2 is locally CAT(0) and geometrically coherent.

The CAT(0) space: coherence

Recall that a connected locally CAT(0) space Y is called *geometrically coherent* if every covering space $X \to Y$ with X connected and $\pi_1(X)$ finitely generated has the following property. For every compact subset $K \subset X$ there is a core C of X containing K.

Let $H \subseteq G$ be finitely generated. $H = \pi_1(X)$ for some connected covering space $X \to Y$. Choose a finite set K of loops representing generators of H. Need to find a core $C \supset K$ in X.

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X can be viewed as a graph of spaces with vertex spaces the components of $p^{-1}(U)$ and edge spaces

$$E = c_1 \times c_2 \times \cdots \times c_p \times \mathbb{R}^q \times \{\frac{1}{2}\},\ p + q = k \geq 0,$$

products of p circles and q lines with $\{\frac{1}{2}\}$, the components of $p^{-1}(T \times \{\frac{1}{2}\})$.

The CAT(0) space: core

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Select $B_E = c_1 \times \cdots \times c_p \times [a_1, b_1] \times \cdots \times [a_q, b_q] \times \{\frac{1}{2}\}$
inside each E so that $K \cap E \subset B_E$.
For every vertex space V of X, we have $\pi_1(V)$ is f.g.; we choose a core $C(V) \supseteq V \cap K$; also, $C(V) \supseteq V \cap B_E$ for every edge E in X.

The core C is the union of the following spaces:

- the cores $C(V)$ of all vertex spaces, and
- all the products B_E in X.

The CAT(0) space: core

Lemma (Bridson and Haefliger, 1999)

Let A be a compact locally CAT(0) metric space. Let X_0 and X_1 be locally CAT(0) metric spaces. If $\varphi_i \colon A \to X_i$ is a local isometry for $i = 0, 1$, then the quotient of $X_0 \coprod (A \times [0,1]) \coprod X_1$ by the equivalence relation generated by $[(a,0) \sim \varphi_0(a); (a,1) \sim \varphi_1(a)]$, for all $a \in A$, is locally CAT(0).

The core C is the union of the cores $C(V)$ of all vertex spaces and of all the products B_E in X.

Corollary

C is locally CAT(0).

C has the same edge groups as X, and $\pi_1(V) = \pi_1(C(V))$, for all V in X. Hence, $\pi_1(C) = \pi_1(X)$.