Hyperplane arrangements, flag complexes and monoid cohomology

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Outline

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- Then we try to put it altogether and state our main results.

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- $|K|$ is the union of the simplices spanned by sets of coordinate vectors corresponding to an element of $\mathcal{F}$.

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- Then $\Delta(P)$ is the barycentric subdivision of K.

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- The nerve of an open cover is fundamental to Čech cohomology.

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- The modern way to formulate his result is via Leray numbers.

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- E.g., $\mathrm{vd}_{\Bbbk}(S^1 \times [0,1]^2) = 2 = \mathrm{vd}_{\Bbbk}(S^1)$.
- $\mathrm{vd}_{\Bbbk}(K)$ is a homotopy invariant.

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- $\mathrm{Ler}_{\Bbbk}(K) = 0$ iff K is a simplex.

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- The clique complex $\mathrm{Cliq}(G)$ is the flag complex with vertex set V and simplices the cliques of G.

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Cliq(G) is 1-representable iff G is chordal and $\overline{G}$ is a comparability graph (Lekkerkerker, Boland).

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- If K is a flag complex, then $I(K)$ is generated by products $x_i x_j$ with $\{x_i, x_j\}$ a non-edge of K^1.
- Such ideals are often called edge ideals since they correspond to edges of the complementary graph of K^1.

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- I believe Fröberg independently discovered the connection between chordal graphs and Leray number 1 in this context.

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Combinatorial objects as LRBs

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- Markov chains on these objects can be analyzed via LRB representation theory.

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Others:

Björner, Athanasiadis-Diaconis, Chung-Graham, . . .

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  "use a book, then put it at the front"
  - orderings of the books $\leftrightarrow$ words containing every letter
  - move book to the front $\leftrightarrow$ left multiplication by generator

Free LRBs and the Tsetlin library

- The free LRB $F(A)$ on a set A consists of all repetition-free words over the alphabet A.
- Product: concatenate and remove repetitions.
- Example: In $F(\{1,2,3,4,5\})$:

$$\mathbf{3} \cdot 145\mathbf{3}2 = \mathbf{3}145\not{\mathbf{3}}2 = \mathbf{3}1452$$

- Tsetlin Library: shelf of books
  "use a book, then put it at the front"
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  - long-term behavior: favorite books move to the front

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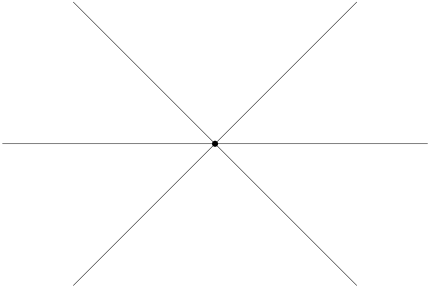
This construction generalizes to matroids and interval greedoids.

Faces of a hyperplane arrangement

A set of hyperplanes partitions $\mathbb{R}^n$ into *faces*:



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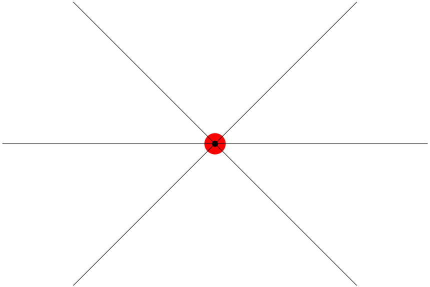

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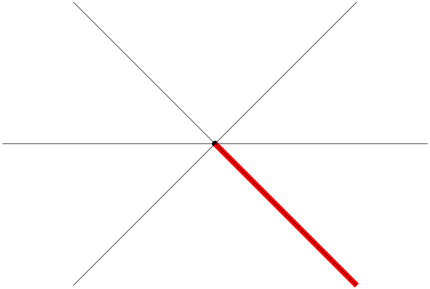

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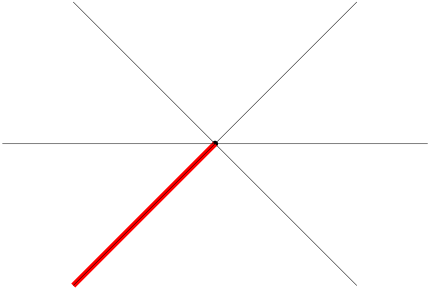

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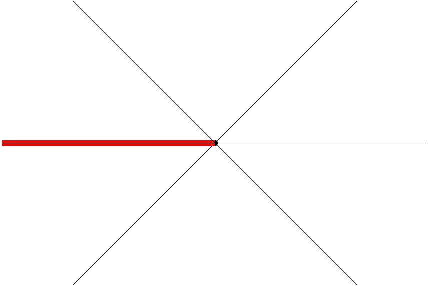

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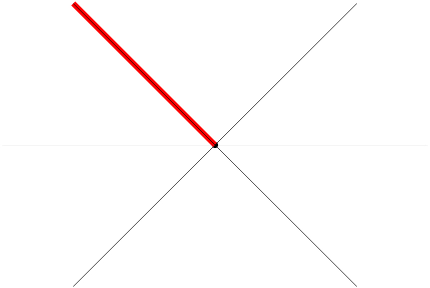

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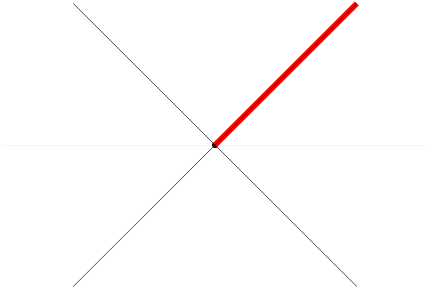

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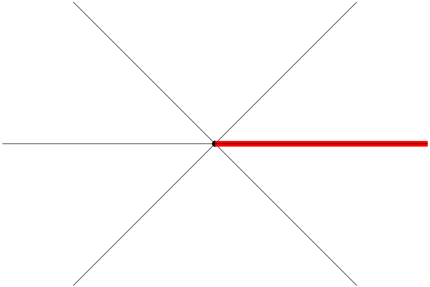

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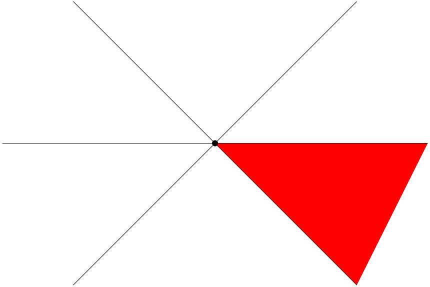

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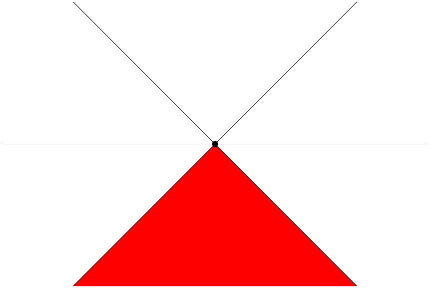

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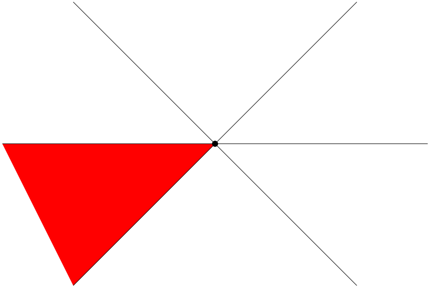

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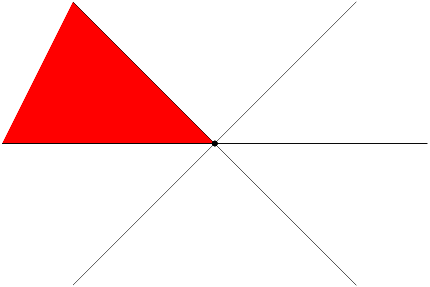

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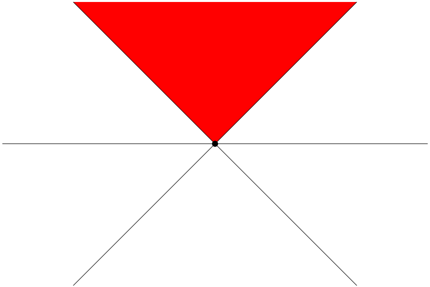

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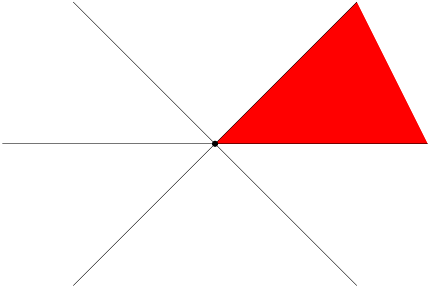

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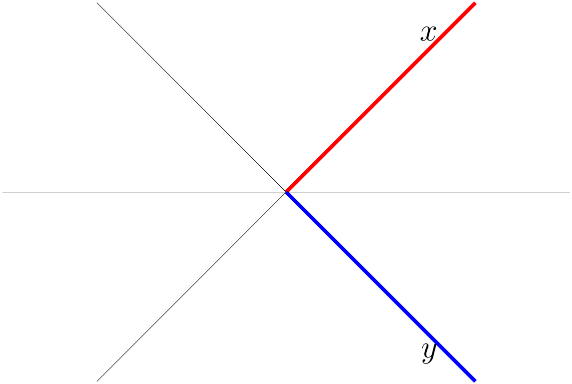
$$xy := \begin{cases} \text{the face first encountered after a small} \\ \text{movement along a line from } x \text{ toward } y \end{cases}$$



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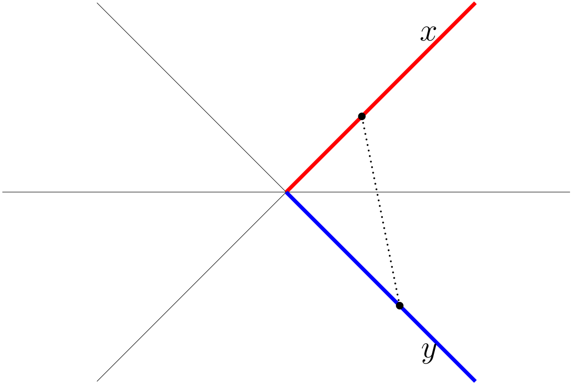

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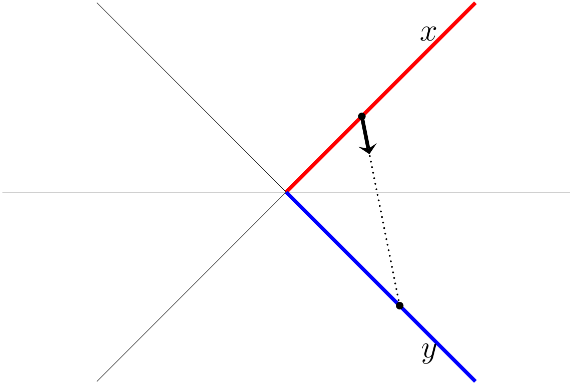

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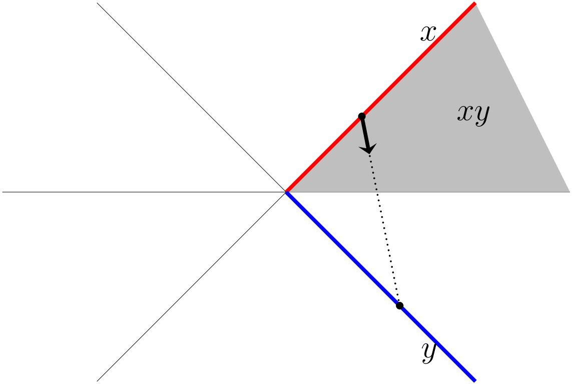

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- For instance, in type A the algebra $\Sigma(W)$ maps onto the character ring with nilpotent kernel.

Free partially commutative LRBs

- The free partially commutative LRB $B(G)$ on a graph $G = (V, E)$ is the LRB with presentation:

$$B(G) = \left\langle V \;\middle|\; xy = yx \text{ for all edges } \{x, y\} \in E \right\rangle$$

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- These are LRB-analogues of free partially commutative monoids and groups.

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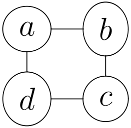
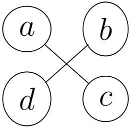
$G =$ 

$\overline{G} =$ 

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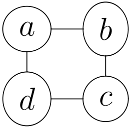
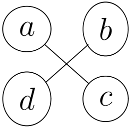
$G =$ 

$\overline{G} =$ 

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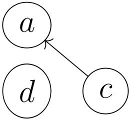

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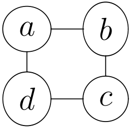
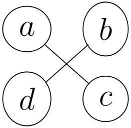
$G =$ 

$\overline{G} =$ 

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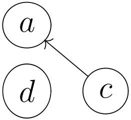
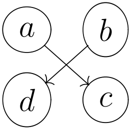


In $B(G)$: $cad = cda = dca$ (c comes before a since $c \to a$)

Random walk on $B(G)$

States: acyclic orientations of the complement $\overline{G}$



Step: left-multiplication by a generator (vertex) reorients all the edges incident to the vertex away from it

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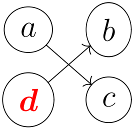

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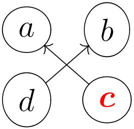

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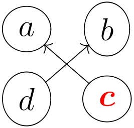

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Athanasiadis-Diaconis (2010): studied this chain using a different LRB (graphical arrangement of $\overline{G}$)

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- Equivalently, $H^n(M; A) = \mathrm{Ext}^n_{\mathbb{Z}M}(\mathbb{Z}, A)$.

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$$\mathrm{cd}_{\Bbbk}(M) = \sup\{n \mid H^n(M; A) \neq 0,\ A \in \Bbbk M\text{-mod}\}.$$

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- We were able to reduce this question to computing $\mathrm{cd}_{\Bbbk}$ of LRBs.

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- Guba and Pride showed that any monoid embeds in a monoid M with $\mathrm{cd}(M) = n$ and $\mathrm{cd}(M^{op}) = m$.
- If B is an LRB, then $\mathrm{cd}(B^{op}) = 0$.

Hyperplane face monoids

Theorem (MSS)

Let $\mathscr{H}$ be an essential hyperplane arrangement in $\mathbb{R}^d$ with corresponding face monoid $\mathcal{F}(\mathscr{H})$. Then $\mathrm{cd}_{\Bbbk}(\mathcal{F}(\mathscr{H})) = d$.

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Corollary

Let $\mathscr{H}$ and $\mathscr{H}'$ be essential hyperplane arrangements in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $M = \mathcal{F}(\mathscr{H}) \times \mathcal{F}(\mathscr{H}')^{op}$. Then $\mathrm{cd}(M) = n$ and $\mathrm{cd}(M^{op}) = m$.

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This theorem applies to free LRBs and LRBs associated to matroids.

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1. $\mathrm{cd}_{\Bbbk}(B(G)) = 0$ *iff* G *is complete;*

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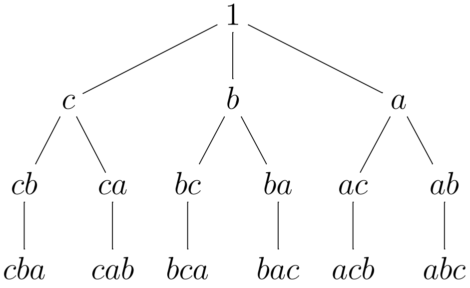

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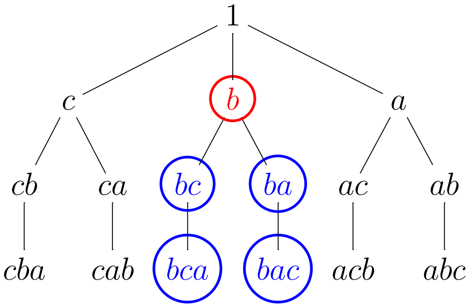

$$F(\{a, b, c\})_{<b} = \{bc, ba, bca, bac\}$$

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- Moreover, each induced subcomplex comes up in this way.

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Recall: $\mathrm{vd}_{\Bbbk}(K) = \min\{d \mid \forall n \geq d, \widetilde{H}^n(K, \Bbbk) = 0\}$.

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Theorem (MSS)

Suppose that $B \curvearrowright K$ is a semi-free action on a contractible simplicial complex. Then the augmented chain complex of K is a projective resolution of $\mathbb{k}$ over $\mathbb{k}B$.

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- The last isomorphism uses that $B_{\geq a}$ is a cone on $B_{>a}$, hence contractible, and the long exact sequence in relative cohomology.

The end

Thank you for your attention!