Rational subsets of wreath products

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Rational sets in arbitrary monoids: Definition 1

Let M be a monoid.

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The set $\mathrm{Rat}(M) \subseteq 2^M$ of all rational subsets of M is the smallest set such that:

- Every finite subset of M belongs to $\mathrm{Rat}(M)$.
- If $L_1, L_2 \in \mathrm{Rat}(M)$, then also $L_1 \cup L_2,\ L_1 L_2 \in \mathrm{Rat}(M)$.
- If $L \in \mathrm{Rat}(M)$, then also $L^* \in \mathrm{Rat}(M)$.

Rational sets in arbitrary monoids: Definition 2

A finite automaton over M is a tuple $A = (Q, \Delta, q_0, F)$ where

- Q is a finite set of states,
- $q_0 \in Q$, $F \subseteq Q$, and
- $\Delta \subseteq Q \times M \times Q$ is finite.

The subset $L(A) \subseteq M$ is the set of all products $m_1 m_2 \cdots m_k$ such that there exist $q_1, \ldots, q_k \in Q$ with

$$(q_{i-1}, m_i, q_i) \in \Delta \text{ for } 1 \leq i \leq k \text{ and } q_k \in F.$$

Then:

$$L \in \mathrm{Rat}(M) \iff \exists \text{ finite automaton } A \text{ over } M : L(A) = L$$

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The rational subset membership problem for G (RatMP(G)) is the following computational problem:

INPUT: A finite automaton A over G and $g \in G$

QUESTION: $g \in L(A)$?

Membership in submonoids/subgroups

The submonoid membership problem for G is the following computational problem:

INPUT: A finite subset $A \subseteq G$ and $g \in G$

QUESTION: $g \in A^*$?

The subgroup membership problem for G (or generalized word problem for G) is the following computational problem:

INPUT: A finite subset $A \subseteq G$ and $g \in G$

QUESTION: $g \in \langle A \rangle$ $(= (A \cup A^{-1})^*)$?

The generalized word problem is a widely studied problem in combinatorial group theory.

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Rips 1982

There are hyperbolic groups with an undecidable subgroup membership problem.

Graph groups

Let (A, E) be a finite undirected graph. The corresponding graph group is $G(A, E) = \langle A \mid ab = ba \text{ for all } (a, b) \in E \rangle$.

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L, Steinberg 2006

The following are equivalent:

- RatMP($G(A, E)$) is decidable
- The submonoid membership problem for $G(A, E)$ is decidable.
- The graph (A, E) does not contain an induced subgraph of one of the following two forms (C4 and P4):

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Open problem

When is the submonoid membership problem for $N_{r,c}$ decidable?

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For the proof, one encodes a tiling problem of the Euclidean plane into the submonoid membership problem for M_2.

Wreath products

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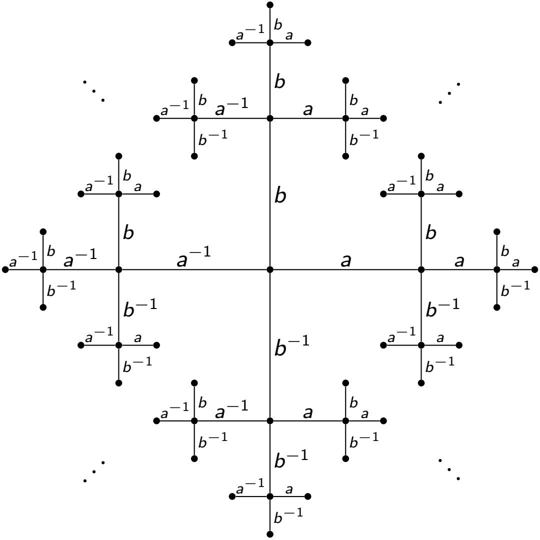
The wreath product $A \wr B$ is the set of all pairs $K \times B$ with the following multiplication, where $(k_1, b_1), (k_2, b_2) \in K \times B$:

$$(k_1, b_1)(k_2, b_2) = (k, b_1 b_2) \text{ with } \forall b \in B : k(b) = k_1(b) k_2(b_1^{-1} b).$$

Wreath product $\mathbb{Z}_2 \wr F(a,b)$ with $\mathbb{Z}_2 = \langle c \mid c^2 = 1 \rangle$

$cbcb^{-1}cabcb^{-1}ca$:



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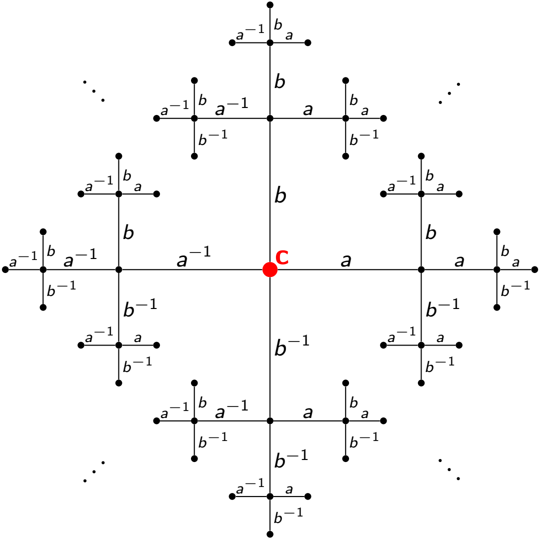

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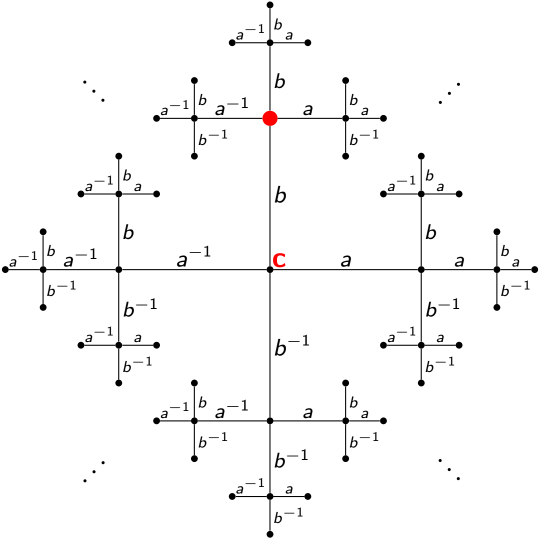

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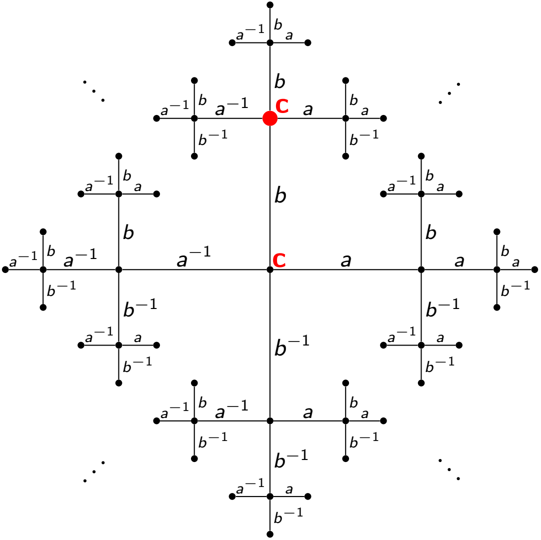

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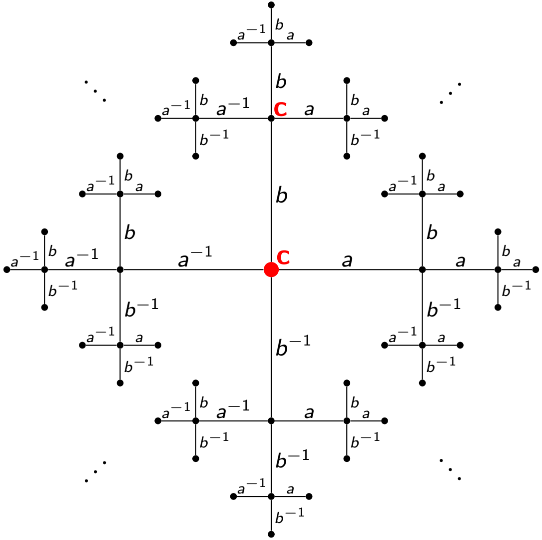

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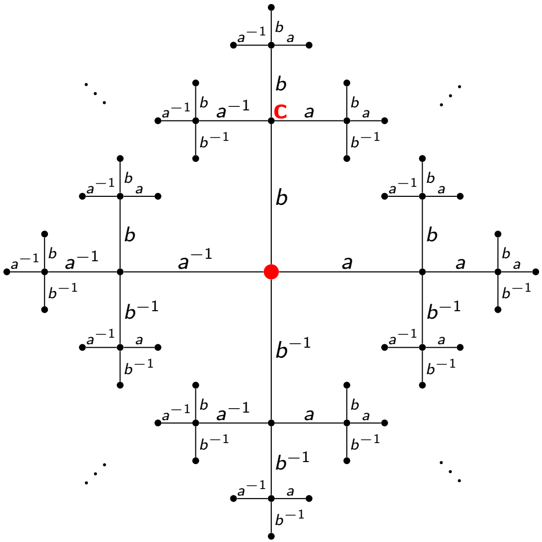

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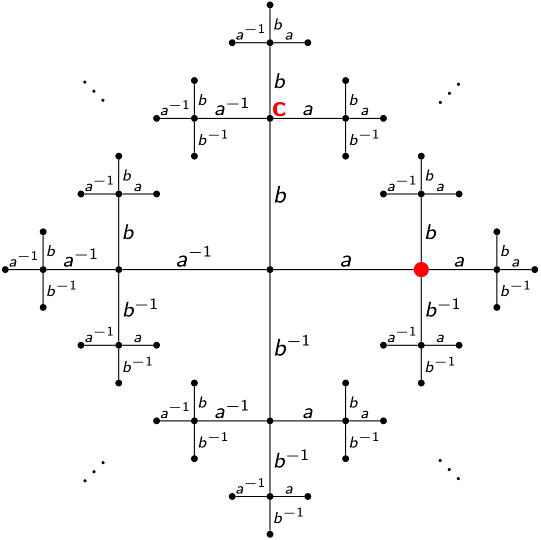

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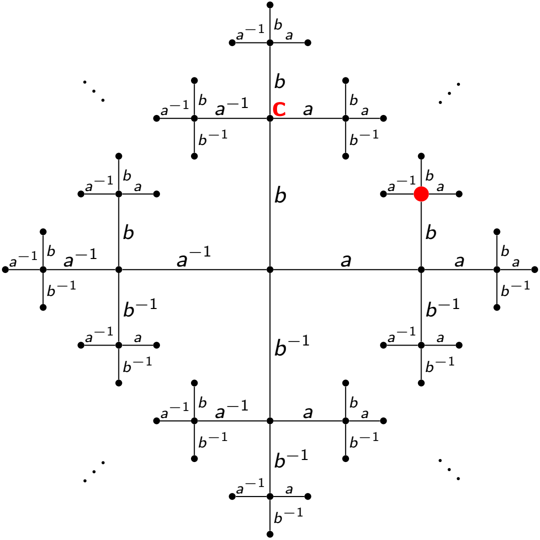

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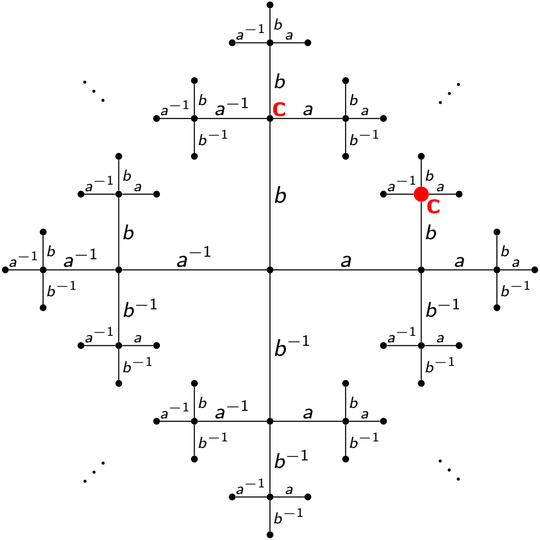

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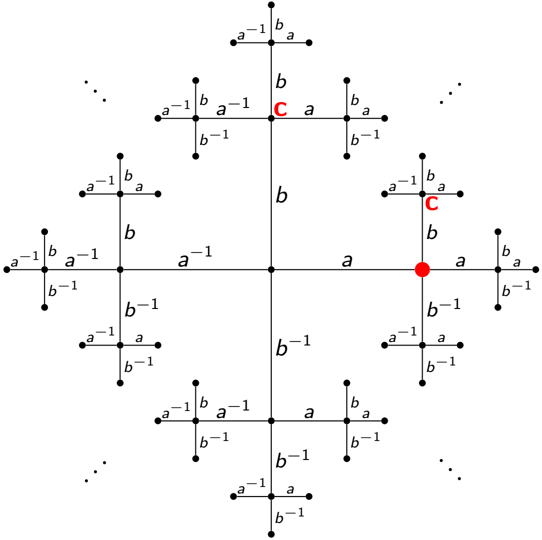

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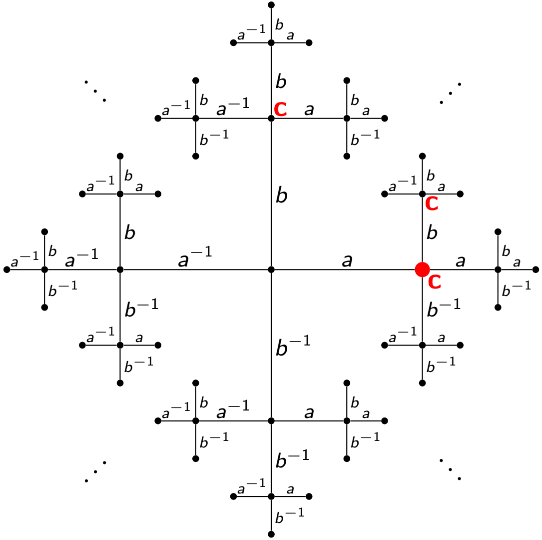

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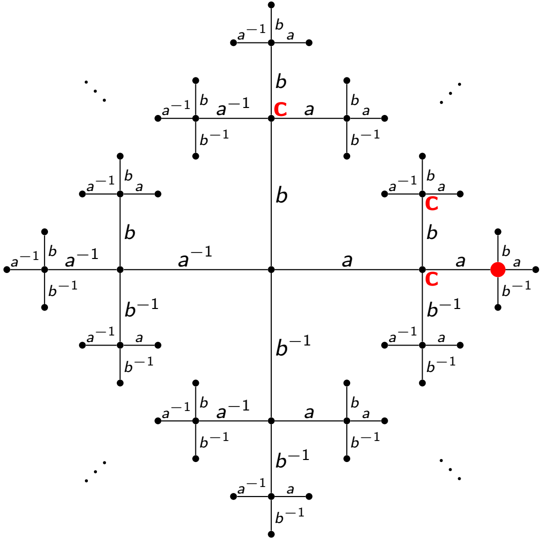

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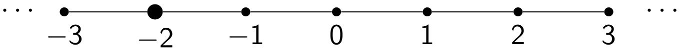

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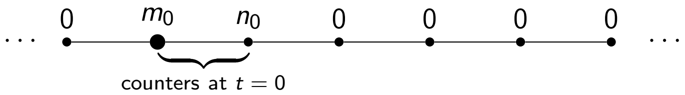

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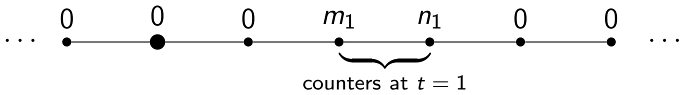

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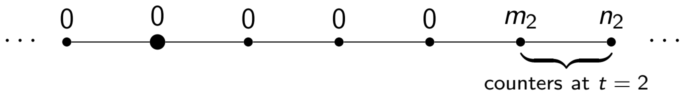

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Fix an automaton $A = (Q, \Delta, q_0, F)$ over the finite alphabet $H \cup \{a, b, a^{-1}, b^{-1}\}$.

We want to check, whether there exists $w \in L(A)$ with $w = 1$ in G.

Loops

Let $p, q \in Q$, $d \in \{a, b, a^{-1}, b^{-1}\}$. A (p, d, q)-loop is an A-path

$$\pi = (p = p_0 \xrightarrow{d} p_1 \xrightarrow{\alpha_1} p_2 \xrightarrow{\alpha_2} p_3 \cdots \xrightarrow{\alpha_{n-1}} p_n \xrightarrow{d^{-1}} p_{n+1} = q)$$

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- $\text{depth}(\pi) = \max\{|u_i| + 1 \mid 1 \leq i \leq n-1\}$
- $\text{effect}(\pi) = d\alpha_1 \cdots \alpha_{n-1} d^{-1} \in K$.

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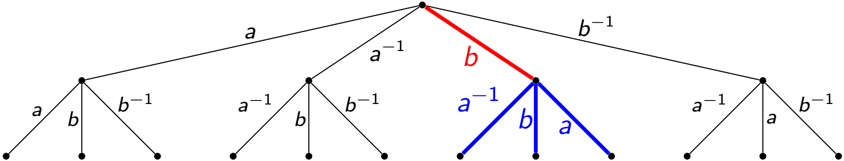

$C_b = \{a^{-1}, b, a\}$

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such that for every $1 \leq i \leq n$ there exists a (p_i, d_i, q_i)-loop π_i with

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We will show:

- P_t is regular and
- an automaton for P_t can be computed.

A well quasi order

A WQO (well quasi order) is a reflexive and transitive relation $\preceq$ (on a set A) such that for every infinite sequence $a_1, a_2, a_3, \ldots$ there exist $i < j$ with $a_i \preceq a_j$.

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For a group H, we define a partial order $\preceq_H$ on X^* (X any finite alphabet) as follows: $u \preceq_H v$ iff there exist factorizations

$$\begin{aligned} u &= x_1 x_2 \cdots x_n \quad (x_i \in X) \\ v &= v_0 x_1 v_1 x_2 \cdots v_{n-1} x_n v_n \end{aligned}$$

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Lemma

For every finite group H, $\preceq_H$ is a WQO.

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This implies that P_t is regular, but can we compute an NFA for P_t?

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For $p, q \in Q$ and $t \in T$ define the regular set

$$R^t_{p,q} = \{(p_0, g_1, p_1)(p_1, g_2, p_2) \cdots (p_{n-1}, g_n, p_n) \in Y_t^* \mid p_0 = p, p_n = q\}.$$

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For $t \in T$, $d \in C_t$, define a regular substitution
$\sigma_{t,d} : X_t \to \mathrm{Reg}(Y_d)$ by

$$\begin{aligned} \sigma_{t,d}(p,d,q) &= \bigcup \{R^d_{p',q'} \mid (p,d,p'),(q',d^{-1},q) \in \Delta\} \\ \sigma_{t,d}(p,u,q) &= \{\varepsilon\} \ \text{ for } u \in C_t \setminus \{d\}. \end{aligned}$$

A fixpoint characterization of P_t

Lemma

$(P_t)_{t \in \{1,a,a^{-1},b,b^{-1}\}}$ is the smallest tuple (w.r.t. to componentwise inclusion) such that for every $t \in \{1, a, a^{-1}, b, b^{-1}\}$ we have $\varepsilon \in P_t$ and

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The lemma follows since $P_t = \bigcup_{i\geq 0} P_t^{(i)}$.

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- Conjecture: Whenevery H is non-trivial and G is not virtually-free, then RatMP($H \wr G$) is undecidable.