Elementary Theory of f.g. nilpotent groups and polycyclic groups

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Outline

Finitely generated groups elementarily equivalent to an arbitrary f.g. nilpotent group.

Groups (not necessarily finitely generated) elementarily equivalent to a free nilpotent group of finite rank.

Classes of polycyclic groups where elementary equivalence and isomorphism are the same.

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Elementary theories

Definition

The elementary theory $Th(\mathcal{A})$ of a group $\mathcal{A}$ (or a ring, or an arbitrary structure) in a language L is the set of all first-order sentences in L that are true in $\mathcal{A}$.

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Two groups (rings) $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent in a language L ($\mathcal{A} \equiv \mathcal{B}$) if $Th(\mathcal{A}) = Th(\mathcal{B})$.

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Tarski type questions

Classification of groups and rings by their first-order properties goes back to A. Tarski and A. Mal'cev.

Elementary classification problem

Characterize in algebraic terms all groups (rings), perhaps from a given class, elementarily equivalent to a given one.

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Nilpotent groups

Definition

A group G is called **nilpotent** if the lower central series:

$$\Gamma_1(G) =_{df} G, \quad \Gamma_{i+1}(G) =_{df} [G, \Gamma_i(G)], i \geq 1,$$

is eventually trivial.

Definition

A group G is called **free nilpotent of rank r and and class c** if

$$G \cong F(r)/\Gamma_{c+1}(F(r)),$$

where $F(r)$ is the free group of rank r.

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Groups elementarily equivalent to a divisible nilpotent group

Definition

If G is torsion free f.g nilpotent group and R is binomial domain then by G^R is meant the P.Hall R-completion of G.

Theorem (Myasnikov, Remeslennikov)

Suppose that G is a torsion free f.g. directly indecomposable nilpotent group and H is a group. Then

$$H \equiv G^{\mathbb{Q}} \Leftrightarrow H \cong G^F,$$

where F is a field so that $\mathbb{Q} \equiv F$. If G is a direct product, then the statement applies to each component of G.

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Elementary equivalence of f.g. nilpotent groups

Notation

- $G' = [G, G]$
- $Is(G') = \{x \in G : x^n \in G',\ \text{for some } n \neq 0\}$

Theorem (A.Myasnikov, 1984)

Let G and H be f.g. nilpotent groups such that $Z(G) \leq Is(G')$. Then

$$G \equiv H \Leftrightarrow G \cong H.$$

Moreover If $K \equiv G \times \mathbb{Z}^n$ then

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As far as we know groups like K in the statement above form the largest class of f.g. nilpotent groups where elementary equivalence implies isomorphism.

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Theorem (F.Oger, 1991)

Suppose that G and H are f.g. nilpotent groups. Then

$$G \equiv H \Leftrightarrow G \times \mathbb{Z} \cong H \times \mathbb{Z}.$$

Combining the above result and the solution of the isomorphism problem for f.g. nilpotent groups by F. Grunewald and D. Segal one can get the following:

There is an algorithm to check whether two f.g. nilpotent groups are elementarily equivalent or not.

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Elementary equivalence and abelian deformations

The result above does not characterize the algebraic structure of f.g. nilpotent groups elementarily equivalent to a given one.

Now we provide such a characterization.

Theorem (A. Myasnikov, M.S.)

Assume G and H are f.g. nilpotent groups. Then $G \equiv H$ if and only if H is an abelian deformation of G.

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Abelian deformation in f.g. nilpotent groups

In general abelian deformations could be defined using symmetric 2-cocyles, however here to make the presentation shorter we avoid using the language of cohomology.
In fact we think that using cohomology and its interaction with the classical theory of covering groups is the best way to understand the abelian deformations (This is precisely how we described our results in the relevant articles).

Abelian deformations via group extensions

Assume G is a f.g nilpotent group so that $Is(G') \cdot Z(G) > Is(G')$ then a group H is an abelian deformation of G if H is isomorphic to a group K such that:

- $G = K$ as sets
- $Is(K') = Is(G')$
- $[x, y]_G = [x, y]_K$
- all the successive quotients in the following series are isomorphic:

$$G > Is(G' \cdot Z(G)) > Is(G') \cdot Z(G) > Is(G')$$

$$K > Is(K' \cdot Z(K)) > Is(K') \cdot Z(K) > Is(K')$$

Zilber's example

Consider the 2-nilpotent presentations:

$$G = \langle a_1, b_1, c_1, d_1 | a_1^5 \in Z(G), [a_1, b_1][c_1, d_1] = 1 \rangle,$$

$$H = \langle a_2, b_2, c_2, d_2 | a_2^5 \in Z(H), [a_2, b_2]^2[c_2, d_2] = 1 \rangle.$$

Theorem (B. Zilber)

$G \equiv H$ but $G \not\cong H$.

Let us first apply a Titze transformation to both G and H to get

$$G \cong \langle a_1, b_1, c_1, d_1, f_1 | f_1 \text{ central}, a_1^5 = f_1, [a_1, b_1][c_1, d_1] = 1 \rangle$$

$$H \cong \langle a_2, b_2, c_2, d_2, f_2 | f_2 \text{ central}, a_2^5 = f_2, [a_2, b_2]^2[c_2, d_2] = 1 \rangle.$$

Now we are going to show that H is an abelian deformation of G. So define a group K by

$$K = \langle a_1, b_1, c_1, d_1, f_1 | f_1 \text{ central}, a_1^5 = f_1^2, [a_1, b_1][c_1, d_1] = 1 \rangle.$$

Then

- $K = G$ as sets (with some cheating)
- $Is(K') = Is(G') = G'$
- $[x, y]_G = [x, y]_K$
-
  - $G/Is(G' \cdot Z(G)) \cong K/Is(K' \cdot Z(K)) \cong \mathbb{Z}^4$
  -

$$\frac{Is(G' \cdot Z(G))}{Is(G') \cdot Z(G)} = \langle a_1, f_1 | a_1^5 = f_1, f_1 = 1 \rangle \cong \frac{\mathbb{Z}}{5\mathbb{Z}}$$
$$\cong \langle a_1, f_1 | a_1^5 = f_1^2, f_1 = 1 \rangle \cong \frac{Is(K' \cdot Z(K))}{Is(K') \cdot Z(K)}$$

  - $Is(G' \cdot Z(G))/Is(G') \cong \langle f_1 \rangle \cong \mathbb{Z} \cong \langle f_1 \rangle \cong Is(K' \cdot Z(K))/Is(K')$.

$$G \cong \langle a_1, b_1, c_1, d_1, f_1 | f_1 \text{ central}, a_1^5 = f_1, [a_1, b_1][c_1, d_1] = 1 \rangle$$
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$$K = \langle a_1, b_1, c_1, d_1, f_1 | f_1 \text{ central}, a_1^5 = f_1^2, [a_1, b_1][c_1, d_1] = 1 \rangle$$

However in K

$$(a_1^3 f_1^{-1})^2 = a_1^6 f_1^{-2} = a_1(a_1^5 f_1^{-2}) = a_1.$$

Now apply the corresponding Titze transformation to the presentation of K to get

$$\begin{aligned} K &= \langle a_1^3 f_1^{-1}, b_1, c_1, d_1, f_1 | f_1 \text{ central}, (a_1^3 f_1^{-1})^5 = f_1, [(a_1^3 f_1^{-1})^2, b_1][c_1, d_1] = 1 \rangle \\ &= \langle a_1^3 = f_1, b_1, c_1, d_1, f_1 | f_1 \text{ central}, (a_1^3 f_1^{-1})^5 = f_1, [a_1^3 f_1^{-1}, b_1]^2[c_1, d_1] = 1 \rangle \\ &\cong \langle a_2, b_2, c_2, d_2, f_2 | f_2 \text{ central}, a_2^5 = f_2, [a_2, b_2]^2[c_2, d_2] = 1 \rangle \\ &\cong H \end{aligned}$$

The statement of the theorem again:

Theorem

Assume G and H are f.g. nilpotent groups. Then $G \equiv H$ if and only if H is an abelian deformation of G.

A few words about the proof

- The action of the ring of integers $\mathbb{Z}$ on every quotient of a central series of G, with the exception of the quotient $Is(G') \cdot Z(G)/Is(G')$, is interpretable in G.
- Therefore there is a definable (set of) Mal'cev basis (bases) in G **almost** all whose commutation and torsion structure constants can be described.
- Only the following abelian extension can not be completely captured in the first order theory of G, and therefore can be deformed as one moves from G to H where $H \equiv G$.

$$1 \to \frac{Is(G') \cdot Z(G)}{Is(G')} \to \frac{Is(G' \cdot Z(G))}{Is(G')} \to \frac{Is(G' \cdot Z(G))}{Is(G') \cdot Z(G)} \to 1$$

- Recall that $Ext(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, i.e. there are exactly p non-equivalent extensions of $\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$ though here only the non-split extensions are possible, i.e. extension of the form:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

Groups elementarily equivalent to unitriangular groups

Theorem (O. Belegradek, 1992)

For a group H one has

$$H \equiv UT_n(\mathbb{Z}) \Leftrightarrow H \ is \ a \ quasi - UT_n(R),$$

some $R \equiv \mathbb{Z}$.

Groups elementarily equivalent to a free nilpotent group

Notation

$$N_{r,c}(\mathbb{Z}) = F(r)/\Gamma_{c+1}(F(r))$$

$$N_{r,c}(R) = \text{ Hall } R\text{-completion of } N_{r,c}(\mathbb{Z}).$$

Theorem (A. Myasnikov, M.S.)

Assume that H is a group. Then

$$H \equiv N_{r,c}(\mathbb{Z}) = G \Leftrightarrow H \text{ is an abelian deformation of } N_{r,c}(R)$$

for some $R \equiv \mathbb{Z}$. Moreover there exists $H \equiv G$ where H is not $N_{r,c}$.

The same argument applies to many other nilpotent groups.

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Why the name abelian deformation?

Assume $\{g_1, \dots, g_r\}$ is a free generating set for $G = N_{r,c}(\mathbb{Z})$. Then

$$C_G(g_j) = \langle g_j \rangle \oplus Z(G).$$

But for a "free" generating set $\{h_1, \dots h_r\}$ of a deformation, $C_H(h_j)$ is an arbitrary abelian extension of $Z(H)$ by $\langle h_j \rangle^R$, i.e. there exists a short exact sequence of abelian groups:

$$1 \to Z(H) \to C_H(h_j) \to \langle h_j \rangle^R \to 1.$$

Elementary Theory of Polycyclic Groups

What do we know?

A counter example due to F. Oger

There are polycyclic groups $G \equiv H$ where $G \times \mathbb{Z} \not\cong H \times \mathbb{Z}$.

Theorem (G. Sabbagh and J. S. Wilson 1991)

If G is polycyclic and H is a f.g. group such that $G \equiv H$ then H is a polycyclic group.

Theorem (D. Raphael 1996)

If $G \equiv H$ are polycyclic groups and $n > 2$ is an integer then there is an embedding $\phi : G \to H$ with $[H : \phi(G)] < \infty$ prime to n.

Note

The theorem does not provide a necessary condition for elementary equivalence.

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Our approach

Definition-Proposition

Every polycyclic group has a unique maximal normal nilpotent subgroup, $Fitt(G)$, called the Fitting subgroup of G.

Theorem (A. Mal'cev)

Every polycyclic group is nilpotent-by-abelian-by finite

Lemma (A. Myasnikov, V. Romankov)

If G is polycyclic group there is a first order formula of the language of groups defining $Fitt(G)$ in G. Moreover the same formula defines $Fitt(H)$ in H for any polycyclic group $H \equiv G$.

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Nilpotent (almost-) supplements

Definition

Assume G is a group and N a normal subgroup. A subgroup C of G is said to be a nilpotent almost-supplement of N in G if $[G : NC] < \infty$.

Theorem (M. L. Newell, D. Segal)

Assume G is a polycyclic group and N a normal nilpotent subgroup of G such that G/N is nilpotent. Then N has a nilpotent almost supplement in G.

Theorem (D. Segal)

Every polycyclic group G has a characteristic "splittable" subgroup of finite index K and therefore a subgroup K where $Fitt(K)$ has a maximal nilpotent supplement in K.

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First-order definable maximal nilpotent supplements

Lemma

Let G be a polycyclic group and N a definable normal subgroup of G with $Z(N) \leq Is(N')$ such that G/N is free abelian. If C is any particular maximal nilpotent supplement of N, such that $Z(C) \leq N$ then C is first-order definable in G with the help of certain constants.

Theorem (A. Myasnikov, M.S.)

Assume G is a polycyclic group satisfying the following conditions and H is a finitely generated group.

(a) The Fitting subgroup $Fitt(G) = N$ of G satisfies $Z(N) \leq Is(N')$.

(b) G/N is free abelian.

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Then $G \equiv H$ if and only if $G \cong H$.

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