Some generalizations of one-relator groups

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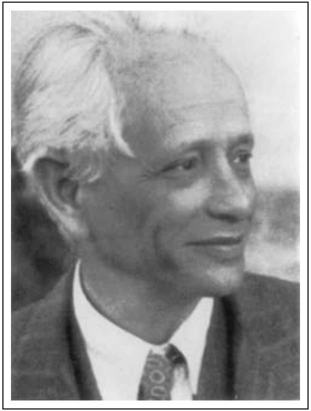
The main topic of this talk is Magnus' induction technique. This is one of the most important techniques to study one-relator quotients of groups. The plan of the talk is the following:

- Recall Magnus induction and its classical applications to one-relator groups.
- Extend Magnus induction to a family of 2-relator groups that include surface-plus-one-relation groups.
- Extend Magnus induction to study one-relation quotients of graph products.

The talk is based in joint work with Warren Dicks, Ramón Flores, Aditi Kar and Peter Linnell.

Outline

Max Dehn



- Max Dehn (Germany 1878-U.S.A. 1952)
- In 1910 publishes a paper with the decision problems.
- He solves the word and conjugacy problem for fundamental groups of orientable surfaces.

Wilhelm Magnus

- Wilhelm Magnus was a student of Dehn who asked him various questions about one-relator groups.
- In 1932 Wilhelm Magnus proved

Theorem

Every one-relator group $\langle x_1, \ldots, x_n \mid r \rangle$ has solvable word problem.

The Freiheitssatz

Let

$$G = \langle x_1, x_2, \ldots, x_d \mid r \rangle,$$

and assume that r is cyclically reduced.

A subset Y of $\{x_1, x_2, \ldots, x_d\}$ is a Magnus subset if it omits some letter of r. A subgroup generated by a Magnus subset is called a Magnus subgroup.

Theorem

- [Freiheitssatz] *A Magnus subset freely generates a subgroup of G.*
- *Moreover, the membership problem for Magnus subgroups in G is solvable.*

Magnus induction (proof of the Freiheitssatz)

Let $G = \langle x_1, x_2, \ldots, x_d \mid r \rangle$.

- There exists a finite chain of one-relator groups

$$G_1 \leqslant G_2 \leqslant \cdots \leqslant G_n = G,$$

  where G_1 is cyclic.
- Each G_i "is" an HNN-extension of G_{i-1}, and the associate subgroups are generated by Magnus subgroups of G_{i-1}.

Example

$$\begin{aligned} \langle x, y \mid x^{-2}yxyxy \rangle &= \langle x, y \mid (x^{-2}yx^2)(x^{-1}yx)y \rangle \\ &= \left\langle x, y, y_1, y_2 \;\middle|\; \begin{array}{c} (x^{-2}yx^2)(x^{-1}yx)y, \\ y_1 = x^{-1}yx, y_2 = x^{-1}y_1x^1 \end{array} \right\rangle \\ &= \langle y, y_1, y_2 \mid y_2y_1y \rangle *_{\langle y, y_1 \rangle} x \end{aligned}$$

Let $G = \langle x_1, x_2, \dots, x_d \mid r \rangle$.

- Word problem.
  Since the membership problem for Magnus subgroups is solvable, the normal form for the HNN extension can be computed.
- Cohomology
  Lyndon used Magnus induction to show that there is a exact sequence of $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z}[G/C_r] \to (\mathbb{Z}G)^d \to \mathbb{Z}G \to \mathbb{Z} \to 0.$$

  Here C_r is the subgroup generated by the root of r.

Outline

Surface groups

- Σ compact, connected surface
- Σ orientable without boundary

$$\pi_1(\Sigma) \cong \langle x_1, \ldots, x_g, y_1, \ldots, y_g \mid [x_1, y_1] \cdot [x_2, y_2] \cdots [x_g, y_g] \rangle.$$

- Σ non-orientable without boundary

$$\pi_1(\Sigma) \cong \langle x_1, \ldots, x_d \mid x_1^2 \cdot x_2^2 \cdots x_d^2 \rangle.$$

- Σ with boundary, then $\pi_1(\Sigma)$ is a free group.

One-relator quotients of $\pi_1(\Sigma)$ are natural generalization of one-relator groups.
Among others, surface-plus-one-relation groups have been studied by Papakyriakopoulos, Hempel, Howie, Bogopolski...

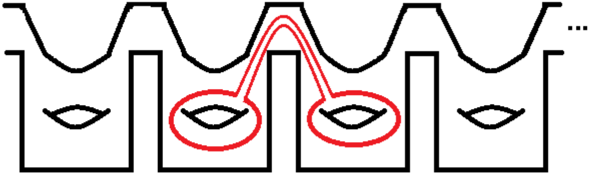
Hempel trick

Hempel trick

- Let Σ be a compact oriented surface and α a simple closed curve on Σ.
- Take a infinite cyclic cover $p\colon \tilde{\Sigma} \to \Sigma$ s.t. every lift of α to $\tilde{\Sigma}$ is a simple closed curve.
- Take a subsurface A of $\tilde{\Sigma}$ containing only one lift $\tilde{\alpha}$ of σ.
- Take a subsurface B of A that does not contain $\tilde{\alpha}$.

We have that

$$\pi_1(\Sigma)/\langle\!\langle \alpha \rangle\!\rangle = \pi_1(A)/\langle\!\langle \tilde{\alpha} \rangle\!\rangle *_{\pi_1(B)} t.$$



Surface-plus-one relation groups fit on Magnus induction

With the Hempel trick one proves that

$$G = \langle\, x_1, \ldots, x_g, y_1, \ldots, y_g \mid [x_1, y_1] \cdot [x_2, y_2] \cdots [x_g, y_g], r \,\rangle$$

is an HNN extension of a one-relator group where the associated subgroups are Magnus subgroups.

In particular one obtain

- G has solvable word problem.
- An exact sequence of $\mathbb{Z}G$ modules

  $$0 \to \mathbb{Z}G \oplus \mathbb{Z}[G/C_r] \to (\mathbb{Z}G)^{2g} \to \mathbb{Z}G \to \mathbb{Z} \to 0.$$

  Here C_r is the subgroup generated by the root of r (in the surface group).

One aims to generalize this construction to non-orientable surfaces.

Hempel groups

Lemma

There exists an isomorphism between $\langle x, y, z \mid [x, y]z^2 \rangle$ *and* $\langle a, b, c \mid a^2b^2c^2 \rangle$.

Let $d \in \mathbb{N}$, let $F := \langle x, y, z_1, \dots, z_d \mid \quad \rangle$, and let u and r be elements of F, such that $u \in \langle z_1, \dots, z_d \rangle$.

We will consider groups of the form

$$\langle x, y, z_1, \dots, z_d \mid [x, y]u = 1, r = 1 \rangle,$$

which generalize one-relator quotients of fundamental groups of surfaces.

Theorem (A,Dicks, Linnell)

Let $G = \langle x, y, z_1, \dots, z_d \mid [x, y]u = 1, r = 1 \rangle$, *where* $u \in \langle z_1, \dots, z_d \rangle$. *Then* G *is either virtually a one-relator group or* G *is an HNN-extension of a one-relator group where the associated subgroups are Magnus subgroups.*

In the latter case we say that a group is a Hempel group.

The proof of the theorem mimics the Hempel trick in an algebraic way. Using automorphism of the form $(x \mapsto xy^{\pm 1}, y \mapsto y)$ and $(x \mapsto x, y \mapsto yx^{\pm 1})$ one can assume that either r has zero x-exponent sum, or $r = x^n$.

Some consequences

- G has solvable word problem.
- Explicit exact sequence of $\mathbb{Z}G$-modules.

More consequences

Hempel groups satisfy the Baum-Connes conjecture by a theorem of Oyono-Oyono.

Computing the Bredon cohomology of Hempel groups, one can describe the equivariant K-theory of $\underline{E}G$.

Theorem (A-Flores)

Let G be a Hempel group $\langle x_1, \dots, x_k \mid w, r \rangle$, with $k \geq 3$, $w \in [x_1, x_2]\langle x_3, \dots, x_k \rangle$. Let $\sqrt[F]{r}$ the root of r (in $\langle x_1, \dots, x_k \mid w \rangle$). Then

- $K_i^G(\underline{E}G) \simeq K_i^{top}(C_r^*(G))$ *for* $i = 0, 1$,
- *there is a split short exact sequence*

$$R_{\mathbb{C}}(G) \to K_0^G(\underline{E}G) \to (\langle r^F \cup w^F \rangle \cap [F, F])/[F, \langle r^F \cup w^F \rangle],$$

- *there is natural isomorphism* $(\langle x_1, \dots, x_k \mid w, \sqrt[F]{r} \rangle)_{ab} \simeq K_1^G(\underline{E}G)$.

Outline

Definitions

- A group is indicable if it is either trivial or maps onto $\mathbb{Z}$.
- A group is locally indicable if all its finitely generated subgroups are indicable.
  Equivalently, any non-trivial finitely generated subgroup has infinite abelianization.
- A group G is effectively locally indicable if it is locally indicable, has solvable word problem, and there exists an algorithm such that given $\{g_1, \dots, g_n\} \subset G - \{1\}$ outputs a map $g_i \to z_i \in \mathbb{Z}$ $i = 1, \dots, n$, that extends to an epimorphism $\langle g_1, \dots, g_n \rangle \to \mathbb{Z}$.

Howie and Brodskii proved independently that torsion-free one-relator groups are locally indicable.

In order to prove this, Howie developed a more powerful inductive argument. Using Howie's induction we proved

Theorem (A-Dicks-Linnell)

Let $G = \langle x, y, z_1, \ldots, z_d \mid [x, y]u = 1, r = 1 \rangle$, where $u \in \langle z_1, \ldots, z_d \rangle$. If G is torsion-free then G is locally indicable.

L^2-Betti numbers

Locally indicability is used to compute the L^2-Betti numbers.

Theorem (A, Dicks, Linnell)

Let G be a surface-plus-one relator group. Then G is of type VFL and, for each $n \in \mathbb{N}$,

$$b_n^{(2)}(G) = \begin{cases} \max\{\chi(G), 0\} = \frac{1}{|G|} & \text{if } n = 0, \\ \max\{-\chi(G), 0\} & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Generalization of Magnus induction to free products

A one-relator group $G = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \mid r \rangle$ can be viewed as a one-relator quotient of two free groups

$$(\langle x_1, \ldots, x_n \mid \rangle * \langle y_1, \ldots, y_m \mid \rangle)/\langle\!\langle r \rangle\!\rangle.$$

We assume that r is not conjugate to an element of one of the factors. The Freiheitssatz can be written as

Theorem (The Freiheitssatz)

Let A and B be free groups and $r \in A * B$ not conjugate to an element of A. Then the map $A \mapsto (A * B)/\langle\!\langle r \rangle\!\rangle$ is injective.

The following was proved by independently by Howie, Brodskii and many others.

Theorem (The locally-indicability Freiheitssatz)

Let A and B be locally indicable and $r \in A * B$ not conjugate to an element of A. Then the map $A \mapsto (A * B)/\langle\!\langle r \rangle\!\rangle$ is injective.

Magnus induction for free products

Let $G = (A * B)/\langle\!\langle r \rangle\!\rangle$ where A and B are locally indicable.

- There exists a finite chain of one-relator quotients of free products of locally indicable groups

$$G_1 \leqslant G_2 \leqslant \cdots \leqslant G_n = G.$$

- G_1 is an amalgamated free product $G_1 = (A_1 * B_1)/\langle\!\langle a = b \rangle\!\rangle$ $a \in A_i$, $b \in B_i$.
- Each G_i "is"

$$(A_i * B_i)/\langle\!\langle r \rangle\!\rangle = A_i *_{A_i'} (A_i' * B_i')/\langle\!\langle r \rangle\!\rangle *_{B_i'} B_i$$

and

$$(A_i' * B_i')/\langle\!\langle r \rangle\!\rangle$$

is an HNN-extension of $G_{i-1} = (A_{i-1} * B_{i-1})/\langle\!\langle r \rangle\!\rangle$, and the associate subgroups are free factors of $A_{i-1} * B_{i-1}$.

Example

Suppose A and B are locally indicable, and $r = a_1 b_1 a_2 b_2$. Then

$$(A * B)/\langle\!\langle r \rangle\!\rangle = A *_{\langle a_1, a_2 \rangle} (\langle a_1, a_2 \rangle * \langle b_1, b_2 \rangle / \langle\!\langle r \rangle\!\rangle) *_{\langle b_1, b_2 \rangle} B.$$

Assume $A = \langle a_1, a_2 \rangle$ and $B = \langle b_1, b_2 \rangle$.

There exists $\phi\colon A \to \mathbb{Z}$. Then $A = \tilde{A} \rtimes \langle t \rangle$, where $\tilde{A} = \ker \phi$, and $t \in A$.

Suppose that $a_1 = t^n \tilde{a}_1$, $a_2 = t^{-n} \tilde{a}_2$.

Put $B^i = t^i B t^{-i}$, $\beta = t^n b_1 t^{-n} \in B^n$ and $\alpha = t^n a_1 t^{-n} \in \tilde{A}$.

Rewriting $a_1 b_1 a_2 b_2$ as $t^n \tilde{a}_1 t^{-n} t^n b_1 t^{-n} \tilde{a}_2 b_2$ and then as $\alpha \beta \tilde{a}_2 b_2$ we see that $(\langle a_1, a_2 \rangle * \langle b_1, b_2 \rangle)/\langle\!\langle r \rangle\!\rangle$ is an HNN-extension of

$$(\tilde{A} * B^0 * \cdots * B^n)/\langle\!\langle \alpha \beta \tilde{a}_2 b_2 \rangle\!\rangle.$$

and the associate subgroups are $\tilde{A} * B^0 * \cdots * B^{n-1}$ and $\tilde{A} * B^1 * \cdots * B^n$.

If one wants to solve the word problem, an effective version of the Magnus induction is needed in order to compute normal forms.

A group G is algorithmically locally indicable if it is efficiently locally indicable and the generalized word problem is solvable in G.

Also the Magnus induction can be generalized to free products amalgamated by a direct factor i.e. groups of the form $(A \times C) *_C (B \times C)$.

Theorem (A-Kar)

Let A, B and C be groups and $G := (A \times C) *_C (B \times C)$. Let $w \in G$ and suppose that it is not conjugate to an element of $A \times C$ nor of $B \times C$. Then the following hold.

(i) *(Freiheitssatz) If A and B are locally indicable, then the natural map $(A \times C) \to G/\langle\!\langle w \rangle\!\rangle$ is injective.*

(ii) *(Membership problem) If moreover A and B are algorithmically locally indicable; then the membership problem for $A \times C$ is solvable in the group $G/\langle\!\langle w \rangle\!\rangle$.*

Corollary

Let A and B be two locally indicable groups and C any group. If $g \in (A * B) \times C$ is not conjugate to an element of $A \times C$ or $B \times C$ then C naturally embeds in the one-relator quotient $((A * B) \times C)/\langle\!\langle g \rangle\!\rangle$.

This gives a "Freiheitssatz" for $F_2 \times F_2 = \langle\, a, b \mid \,\rangle \times \langle\, c, d \mid \,\rangle$.

If $g \in F_2 \times F_2$ is not conjugate to an element of $\langle\, X \,\rangle$, where $|X| = 3$, $X \subseteq \{a, b, c, d\}$ then $\langle\, X \,\rangle \cap \langle\, g^{F_2 \times F_2} \,\rangle = \emptyset$.

Outline

Graph products

Graph product

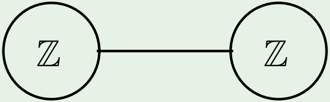
1. A graph product $\Gamma\mathfrak{G}$ is a group codified by a graph Γ and a family of groups $\mathfrak{G} = \{G_v : v \in V\Gamma\}$ indexed by the vertices of Γ.
2. $\Gamma\mathfrak{G}$ is the quotient of $*_{v \in V\Gamma} G_v$ by the relation:

$$[g, h] = 1$$

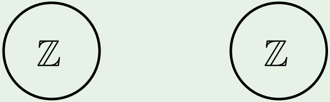
For every $g \in G_u$, $h \in G_v$, $u \neq v$ and (u, v) is an edge of Γ.

Example

$$\mathbb{Z}^2 = \langle\, a, b \mid ab = ba \,\rangle$$

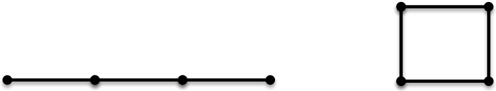


$$\mathbb{F}_2 = \langle\, a, b \mid \ \rangle$$



Definition

A graph L is called starred if it is finite and has no incidence of full subgraphs isomorphic to either L_3, the line of length three, or C_4, the cycle of length 4.



Every graph product over an starred graph is either a free product or a direct product.

Direct Products

Theorem

Let A and B be two finitely presented groups and $a \in A$ and $b \in B$ such that the word problem is solvable for $A/\langle\!\langle a \rangle\!\rangle$ and $B/\langle\!\langle b \rangle\!\rangle$. Then the word problem for $G = (A \times B)/\langle\!\langle (a,b) \rangle\!\rangle$ is solvable.

Proof.

The group $G = A \times B/\langle\!\langle (a,b) \rangle\!\rangle$ fits into the following exact sequence.

$$1 \to \frac{\langle a^A \rangle \times \langle b^B \rangle}{\langle (a,b)^{A\times B} \rangle} \to G \xrightarrow{\pi} \frac{A}{\langle a^A \rangle} \times \frac{B}{\langle b^B \rangle} \to 1$$

There is a natural epimorphism ϕ,

$$\frac{\langle a^A \rangle}{\langle [a,A] \rangle} \times \frac{\langle b^B \rangle}{\langle [b,B] \rangle} \xrightarrow{\phi} \frac{\langle a^A \rangle \times \langle b^B \rangle}{\langle (a,b)^{A\times B} \rangle} = \ker(\pi).$$

Hence $\ker(\pi)$ is abelian of rank at most 2. Then G is fin pres and an extension of two groups with solvable word problem. □

Starred graphs products of poly-Z.

Let G be a graph product of poly-$\mathbb{Z}$ groups over an starred graph. Then

- $G = (A \times C) *_C (B \times C)$.
  Moreover, A, B, C are graph products over starred graphs with fewer vertices.
- $A \times C$ and $B \times C$ have solvable generalized word problem This follows from a theorem of Kapovich, Myasnikov and Wiedmann.
- poly $\mathbb{Z}$-groups are effectively locally indicable.
- graph products of effective locally indicable groups is effective locally indicable this can be seen by induction on the number of vertices using and undertanding the short exact sequence we obtain when quotient by the normal closure of a vertex groups.

Theorem (A-Kar)

Let Γ be a starred graph and $\mathfrak{G}$ be a family of poly- (infinite cyclic) groups. Let $g \in G = \Gamma\mathfrak{G}$. Then, the word problem of the one-relator quotient $G/\langle g^G \rangle$ is solvable.

Thank You!