On the nilpotent genus of groups

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Outline

Nilpotent Genus: 4 Problems of Baumslag

If each finite subset of a group Γ injects into some nilpotent (or finite) quotient of Γ, then it is reasonable to expect that one will be able to detect many properties of Γ from the totality of its nilpotent (or finite) quotients. The extent to which this is true has been examined repeatedly.

Defn

A group G is residually nilpotent if every non-trivial element of Γ survives in some nilpotent quotient of G, equivalently

$$\bigcap_n \gamma_n(G) = 1.$$

Baumslag says that two finitely generated groups have the same nilpotent genus if both are residually nilpotent, and they have the same lower central sequence, equivalently, the same nilpotent quotients (up to isomorphism). A res. nilp. group in the nilpotent genus of a free group is termed parafree. If G is fin gen and residually nilpotent, then it is residually finite

Some Residually Nilpotent Groups

Magnus (1935) proved that finitely generated free groups are residually nilpotent, hence so are finitely generated residually free groups (for example the subdirect products of free and surface groups).

The direct product of n copies of the free group of rank 2 can be regarded as the right angled Artin group (RAAG) associated to the simplicial join of n copies of the 0-sphere.

Recall that the RAAG associated to the finite graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E \subset V \times V$ is given by the presentation

$$A = \langle a_1, \ldots, a_n \mid [a_i, a_j] = 1 \text{ iff } (i,j) \in E \rangle.$$

Theorem (Duchamp and Krob, 1992)

RAAGs are residually torsion-free nilpotent.

Around parafree groups

Γ is parafree if it is residually nilpotent and same genus as some free group. Baumslag produces many examples of finitely presented parafree groups that are not free, but the nature of parafree groups remains a mystery.

$$G_{ij} = \langle a, b, c \mid a = [c^i, a].[c^j, b] \rangle.$$

cf: **Open problem:** Does there exist a finitely generated residually finite group Γ that has the same finite quotients as some free group, i.e. $\hat{\Gamma} \cong \hat{F}_r$

$$\hat{\Gamma} := \lim_{\leftarrow} \Gamma/N \qquad |\Gamma/N| < \infty$$

is the profinite completion of Γ.

Theorem (B,Conder, Reid)

If Γ is a Fuchsian group (e.g. a free group) and Λ is a lattice in a connected Lie group with $\hat{\Gamma} \cong \hat{\Lambda}$, then $\Gamma \cong \Lambda$.

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Propn (B,Reid)

A non-free parafree group cannot be a lattice in a connected Lie group.

There **do** exist lattices in $\mathrm{PSL}(2, \mathbb{C})$ that are *virtually* residually nilpotent and have the same nilpotent quotients as a free group.

homology boundary links

$L \subset S^3$ of m components is called a *homology boundary link* if there exists an epimorphism $h : \pi_1(S^3 \setminus L) \to F$ where F is a free group of rank m. An old theorem of Stallings implies that this map induces an isomorphism of lower central series quotients.

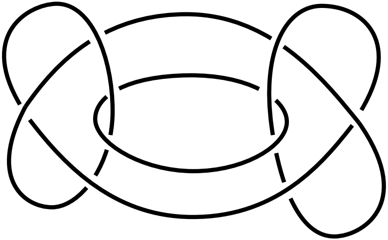

Figure: A boundary homoloy link

Baumslag: How diverse can groups with same genus be?

Problem

∃? finitely generated, residually (t.f.) nilpotent groups, same nilpotent genus, one finitely presented and the other is not?

Problem

∃? finitely presented, residually (t.f.) nilpotent groups, same nilpotent genus, s.t. one has a solvable conjugacy problem and the other does not?

Problem

∃? finitely presented, residually (t.f.) nilpotent groups, same nilpotent genus, one with finitely generated $H_2(-, \mathbb{Z})$ and the other not?

Problem

∃? Let G be a finitely generated parafree group and let $N < G$ be a finitely generated, non-trivial, normal subgroup. Must N be of finite index in G?

Answers [B, Reid]

Theorem

1. $\exists$ *finitely generated, residually t.f. nilpotent groups* $H \hookrightarrow D$ *of the same nilpotent genus s.t.* D *is finitely presented and* H *is not.*
2. $\exists$ *finitely presented, residually t.f. nilpotent groups* $P \hookrightarrow \Gamma$ *of the same nilpotent genus s.t.* Γ *has a solvable conjugacy problem and* P *does not.*
3. $\exists$ *finitely generated, residually t.f. nilpotent groups* $N \hookrightarrow \Gamma$ *of the same nilpotent genus s.t.* $H_2(\Gamma, \mathbb{Z})$ *is finitely generated but* $H_2(N, \mathbb{Z})$ *is not.*
4. *Let* G *be a finitely generated parafree group, and let* $N < G$ *be a non-trivial normal subgroup. If* N *is finitely generated,* G/N *is finite.*

(3) is connected to (but does not solve) the *parafree conjecture*, which asserts that the second homology of a parafree group should be trivial.

I'll say a little about the idea of the proofs of (1) and (2), and more about (3). To prove (4) one considers ℓ_2-betti numbers and $\widehat{G}_{\rm nil}$.

Warm up: Bridson-Grunewald [2006]

Question

If Γ is residually finite, what can one tell about it from it's set of finite homomorphic images, i.e. from its actions on all finite sets?

$$\hat{\Gamma} := \lim_{\leftarrow} \Gamma/N, \qquad |\Gamma/N| < \infty.$$

Question (∃? Grothendieck Pairs (1970))

If Γ_i are fp, residually finite and $u : \Gamma_1 \hookrightarrow \Gamma_2$ induces an isomorphism $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$, must $u : \Gamma_1 \to \Gamma_2$ be an isomorphism?

Pro-Nilpotent Completions

Baumslag's questions can usefully be rephrased in terms of

$$\hat{G}_{\rm nil} := \lim_{\leftarrow} G/\gamma_n(G).$$

Propn (B, Reid)

$u : P \hookrightarrow \Gamma$ f.g., residually finite groups, $u_c : P/P_c \to \Gamma/\Gamma_c$ induced map. If $\hat{u}$ is iso, then u_c is iso for all $c \geq 1$, and

$$\widehat{u}_{\rm nil} : \widehat{P}_{\rm nil} \overset{\cong}{\to} \widehat{\Gamma}_{\rm nil}$$

. In particular, if P and Γ are residually nilpotent, then they have the same nilpotent genus.

IDEA: Construct Grothendieck pairs with differing properties and try to make them residually nilpotent (e.g. by embedding them in RAAGs).

Designer Groups, Rips and 1-2-3 Thm

$\exists$ algorithm with input a finite, aspherical presentation $\mathcal{Q}$ and output a FINITE presentation (by [BBMS]) for the fibre-product

$$P := \{(\gamma_1, \gamma_2) \mid p(\gamma_1) = p(\gamma_2)\} \subset H \times H$$

associated to a s.e.s. (given by [Rips])

$$1 \to N \to H \xrightarrow{p} Q \to 1$$

N 2-generator, H small cancellation, $Q = |\mathcal{Q}|$ (to be cunninging invented).

"1-2-3 Thm" refers to fact that N, H and Q are of type $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ respectively. [Baumslag, B, Miller, Short]

Refinements (B-Haefliger, Wise, Haglund-Wise) place more stringent conditions on H, e.g. locally CAT(-1) or virtually special.

Solving Grothendieck's Problem

Question

$u : \Gamma_1 \hookrightarrow \Gamma_2$ *fp, rf,* $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ *iso, is* $u : \Gamma_1 \to \Gamma_2$ *iso ???*

Grothendieck: yes in many cases, e.g. arithmetic groups. Platonov-Tavgen (later Bass–Lubotzky, Pyber): no for certain finitely generated groups.

Theorem (after B-Grunewald)

$\exists$ *hyperbolic, virtually special* H *and finitely presented subgroup* $P \hookrightarrow \Gamma := H \times H$ *of infinite index, such that* P *is not abstractly isomorphic to* Γ*, but the inclusion* $u : P \hookrightarrow \Gamma$ *induces an isomorphism* $\hat{u} : \hat{P} \to \hat{\Gamma}$*.*

PROOF: Build Q with finite aspherical presentation, no finite quotients and $H_1(Q) = H_2(Q) = 0$ Apply the virtually special Rips construction. Use 1-2-3 Theorem to deduce that the fibre product is finitely presented. Prove $\hat{P} \to \hat{H} \times \hat{H}$ is isomorphism

Solving the first of Baumslag's problems

Crudely speaking, one exploits the above constructions, using the virtually special nature of the hyperbolic groups in a suitable Rips construction to ensure that all of the groups are residually torsion-free nilpotent.

$$1 \to N \to H \to Q \to 1.$$

If one inputs a group Q that has an apsherical presentation and unsolvable word problem, then the fibre product $P < H \times H$ has an unsolvable conjugacy problem.

Baumslag's H_2 problem

Theorem (B-Reid, 2012)

There exists a pair of finitely generated residually torsion-free nilpotent groups $N \hookrightarrow \Gamma$ that have the same nilpotent genus and the same profinite completion, but Γ is finitely presented while $\dim H_2(N, \mathbb{Q}) = \infty$.

A Rips construction followed by spectral sequence calculations, with input a designer group $Q = \tilde{\Delta}$.

Proposition

$\exists$ a torsion-free, finitely presented group $\tilde{\Delta}$ with no non-trivial finite quotients, $H_1(\tilde{\Delta}, \mathbb{Z}) = H_2(\tilde{\Delta}, \mathbb{Z}) = 0$ *and* $\dim H_3(\tilde{\Delta}, \mathbb{Q}) = \infty$.

The group $\tilde{\Delta}$

$$B_p = \langle a_1, a_2, b_1, b_2 \mid$$
$$a_1^{-1} a_2^p a_1 a_2^{-p-1},\ b_1^{-1} b_2^p b_1 b_2^{-p-1},\ a_1^{-1}[b_2, b_1^{-1} b_2 b_1],\ b_1^{-1}[a_2, a_1^{-1} a_2 a_1]\rangle$$

The salient features of B_p are that it is finitely presented, acyclic over $\mathbb{Z}$, has no finite quotients, contains a 2-generator free group, F say, and is torsion-free.

$$\Delta := (A \times A) *_S (A \times A),$$

the double of $A \times A$ along $S < F \times F$, where $S = \ker(F \times F \to \mathbb{Z})$. The universal central extension $\tilde{\Delta}$ is the group we seek.

Baumslag's Fourth Problem

Theorem (B, Reid)

G fin gend parafree, $1 \neq N < G$ normal.

$$N \quad \text{f.g., then} \quad |G/N| < \infty.$$

This is different to the other results: previously we were proving that groups in the same genus could be quite different, now we're proving they have something in common.

The method of proof is also very different: it uses the structure of pro-p groups and L^2-betti numbers.

L^2-betti numbers

Analytic definition, but Lück's Approximation Theorem: Γ *finitely presented* and

$$\Gamma = N_1 > N_2 > \ldots > N_m > \ldots,$$

normal, finite index, $\bigcap_m N_m = 1$; Lück proves

$$\lim_{m\to\infty} \frac{b_1(N_m)}{[\Gamma : N_m]} = b_1^{(2)}(\Gamma).$$

For finitely generated G,

$$\limsup_{m\to\infty} \frac{b_1(N_m)}{[\Gamma : N_m]} \leq b_1^{(2)}(\Gamma).$$

$b_1^{(2)}(F_r) = r - 1$ for F_r free of rank r.

Theorem (Gaboriau)

$$1 \to N \to \Gamma \to \Lambda \to 1$$

N fin gen, Λ infinite, then $b_1^{(2)}(\Gamma) = 0$.

L^2-betti numbers and nilpotent quotients

Theorem

$1 \neq N \lhd \Gamma$ with N, Γ fin gen'd. Let F be finitely pres'd group, residually-p for some prime p. If there is an injection $\Gamma \hookrightarrow \widehat{F}_p$ with dense image and $b_1^{(2)}(F) > 0$, then $|\Gamma/N| < \infty$.

Idea of proof: one can see the torsion-free rank of the abelianisation of subgroups of p-power index by examining the pro-p completion $\widehat{F}_p$. By examining the induced topology on finite index subgroups carefully, one gets an inequality which works well in the 1-sided version of Lück's Theorem.

We want to settle Baumslag's 4th problem by applying this result with F a free group and Γ a parafree group. We can do so because free nilpotent groups are residually-p for all primes p (Gruenberg, 1957), and since Γ is parafree, it is isomorphic to a dense subgroup in the pro-nilpotent and pro-p completions of some F.