Tits alternatives for graph products

Ashot Minasyan
(Joint work with Yago Antolín)

University of Southampton

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Background and motivation

Theorem (J. Tits, 1972)

Let H be a finitely generated subgroup of $\mathrm{GL}_n(F)$ for some field F. Then either H is virtually solvable or H contains a non-abelian free subgroup.

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Recall: a group G is large is there is a finite index subgroup $K \leqslant G$ s.t. K maps onto $\mathbb{F}_2$.

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H contains a copy of $\mathbb{F}_2$.

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The thm. of Noskov-Vinberg implies that Coxeter gps. satisfy the Strong Tits Alternative rel. to $\mathcal{C}_{vab}$.

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$$[a, b] = 1 \ \forall a \in G_u, \forall b \in G_v \text{ whenever } (u, v) \in E\Gamma.$$

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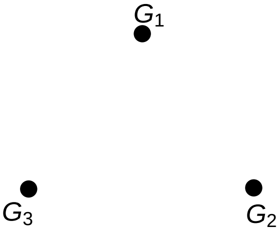

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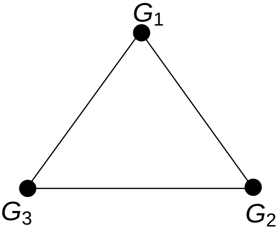

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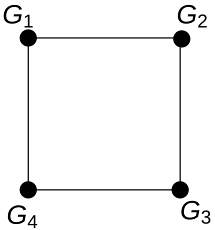

Figure : $\Gamma\mathfrak{G} \cong (G_1 * G_3) \times (G_2 * G_4)$

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Special subgroups

If $A \subseteq V\Gamma$ and Γ_A is the full subgraph of Γ spanned by A then $\mathfrak{G}_A := \{G_v \mid v \in A\}$ generates a special subgroup G_A of $G = \Gamma\mathfrak{G}$ which is naturally isomorphic to $\Gamma_A\mathfrak{G}_A$.

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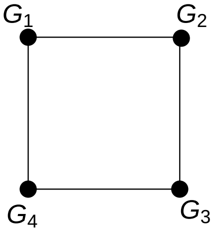

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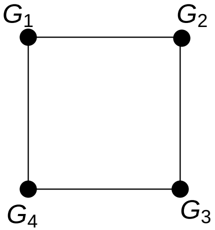

Figure : $G_{\{1,2\}} \cong G_1 \times G_2$, $G_{\{1,3\}} \cong G_1 * G_3$

There is a natural retraction $\rho_A : G = \Gamma\mathfrak{G} \to G_A$ defined by

$$\rho_A(g) = \begin{cases} g & \text{if } g \in G_u \text{ for some } u \in A \\ 1 & \text{if } g \in G_v \text{ for some } v \in V\Gamma \setminus A \end{cases}$$

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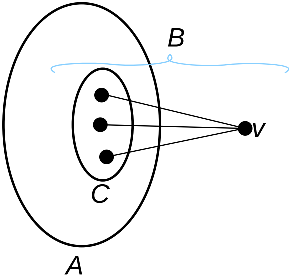

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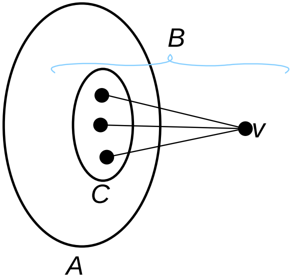

then

$$G = G_A *_{G_C} G_B \text{ and } G_B \cong G_C \times G_v.$$

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Let $\mathcal{C}$ be a class of gps. with (P0)–(P4). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

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Evidently the conditions (P0)–(P4) are necessary.

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Corollary

If all vertex gps. are linear then $G = \Gamma\mathfrak{G}$ satisfies the Tits Alternative rel. to $\mathcal{C}_{vsol}$.

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(P5) is necessary, b/c if $L \neq \{1\}$ has no proper f.i. sbgps., then $L * L$ cannot be large.

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Corollary

Suppose $\mathcal{C} = \mathcal{C}_{sol-m}$ for some $m \geq 2$ or $\mathcal{C} = \mathcal{C}_{vsol-n}$ for some $n \geq 1$. Let G be a graph product of gps. from $\mathcal{C}$. Then any f.g. sbgp. of G either belongs to $\mathcal{C}$ or is large.

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Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Strongest Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H maps onto $\mathbb{F}_2$.

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Observe that if $L * L$ maps onto $\mathbb{F}_2$ then L must have an epimorphism onto $\mathbb{Z}$.

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Theorem (Baudisch, 1981)

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Combining with a result of Lyndon-Schützenberger we also get

Corollary

If G is a RAAG and $a, b, c \in G$ satisfy $a^m b^n = c^p$, for $m, n, p \geq 2$, then a, b, c pairwise commute.

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Thus we can suppose that

(2) $$\rho_A(H) \text{ is abelian for every } A \subsetneqq V\Gamma.$$

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If Γ is reducible then $V\Gamma = A \sqcup B$ and $G = G_A \times G_B$. Thus $H \leqslant \rho_A(H) \times \rho_B(H)$ is abelian by (2).

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If G is a RAAG corresponding to a finite irreducible graph Γ with $|V\Gamma| \geq 2$ and $H \leqslant G$ then one of the following holds:

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Hence

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Recall that $G = G_A *_{G_C} G_B$. By gen. Kurosh Thm. (3) $\Longrightarrow$

$$H = H_1 * \cdots * H_k * F,$$

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Recall that $G = G_A *_{G_C} G_B$. By gen. Kurosh Thm. (3) $\Longrightarrow$

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where F is free, and $H_i \leqslant g_i G_A g_i^{-1}$ or $H_i \leqslant g_i G_B g_i^{-1}$ for some $g_i \in G$.

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Since each H_i maps onto $\mathbb{Z}$ (follows from (P6)), we deduce that H maps onto $\mathbb{Z} * \mathbb{Z} \cong \mathbb{F}_2$. □