Uniform Words are Primitive

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Outline

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- Uniform words, Primitive words

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- Some ingredients of the proof

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$w \in \mathbf{F}_k$ is called **uniform** *if $\forall$ finite group G, $\alpha_G(w) \sim U(G)$.*

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- Primitive words are rare.

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w is primitive $\Longrightarrow$ w is uniform

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- $\exists$ many similar and extended open problems

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Theorem 1 (2011)

The conjecture holds for $\mathbf{F}_2$.

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Theorem 2 (P-Parzanchevski)

The conjecture holds for $\mathbf{F}_k$ $\forall k$.

- *The proof involves: Stallings core graphs, random covering spaces, Möbius inversions, algebraic extensions of free groups,...*

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Thm 3 (the last statement implies "w is primitive")

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- w not primitive in $\mathbf{F}_k \Longrightarrow \pi(w) \leq k$
- Thus, $\pi(w) \in \{0, 1, 2, \ldots, k\} \cup \{\infty\}$

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E.g. $\pi([a, b]) = \pi(a^2 b^2) = 2$

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E.g. $\pi([a, b]) = \pi(a^2 b^2) = 2$

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Fixed points in S_n - The Key Result

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3. 2 new criteria (& algos) to detect primitivity (& free factors)

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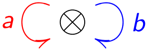

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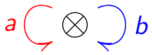

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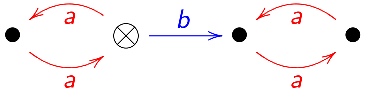

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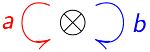

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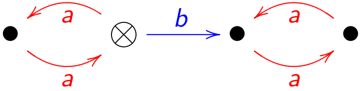

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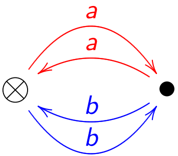

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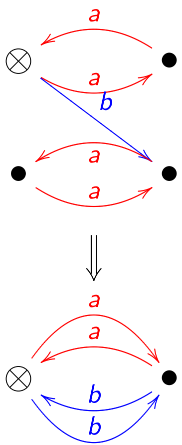

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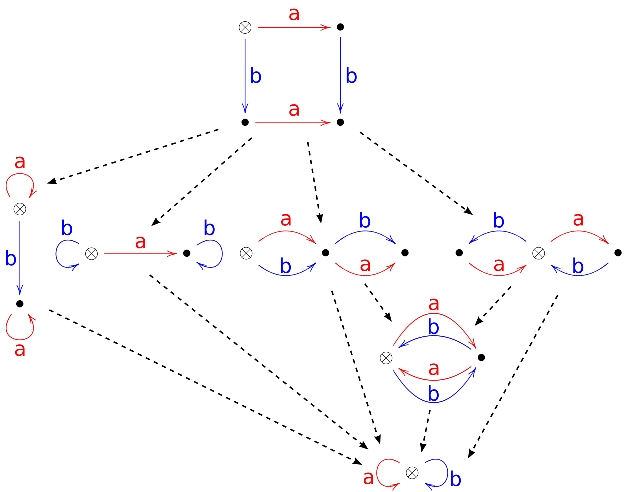

Ingredients of the Proof (cont.)

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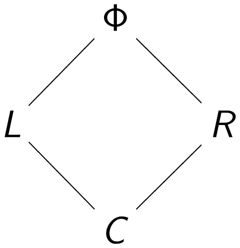

Ingredients of the Proof (cont.)

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Ingr. 4: Random Coverings of Core Graphs

Ingredients of the Proof (cont.)

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Geom. interpretation of $\Phi_{H,J} = \mathbb{E}\left|\overset{\text{comm}}{\text{fix}}\left(\alpha_{J,S_n}(H)\right)\right|$

Ingredients of the Proof (cont.)

Ingr. 4: Random Coverings of Core Graphs

Geom. interpretation of $\Phi_{H,J} = \mathbb{E}\left| \overset{\text{comm}}{\text{fix}} \left(\alpha_{J,S_n}(H)\right)\right|$

$\mathrm{Hom}(J, S_n)$

Ingredients of the Proof (cont.)

Ingr. 4: Random Coverings of Core Graphs

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$$\text{Hom}(J, S_n) \longleftrightarrow \begin{matrix} J - \text{set} \\ \text{structures} \\ \text{on}\{1..n\} \end{matrix}$$

Ingredients of the Proof (cont.)

Ingr. 4: Random Coverings of Core Graphs

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Ingredients of the Proof (cont.)

Ingr. 4: Random Coverings of Core Graphs

Geom. interpretation of $\Phi_{H,J} = \mathbb{E}\left|\ {}^{\text{comm}}_{\text{fix}}\ (\alpha_{J,S_n}(H))\right|$

$$\begin{matrix} \color{red}{\text{random}} \\ \text{Hom}(J, S_n) \end{matrix} \longleftrightarrow \begin{matrix} \color{red}{\text{random}} \\ J-\text{set} \\ \text{structures} \\ \text{on}\{1..n\} \end{matrix} \longleftrightarrow \begin{matrix} \color{red}{\text{random}} \\ n-\text{coverings} \\ \text{of } \Gamma(J) \text{ with} \\ p^{-1}(\otimes) = \{1..n\} \end{matrix}$$

Ingredients of the Proof (cont.)

Ingr. 4: Random Coverings of Core Graphs

Geom. interpretation of $\Phi_{H,J} = \mathbb{E}\left|\overset{\text{comm}}{\text{fix}}\left(\alpha_{J,S_n}(H)\right)\right|$

$$\begin{array}{ccccc} & & \text{random} & & \text{random} \\ \text{random} & & J-\text{set} & & n-\text{coverings} \\ \text{Hom}(J,S_n) & \longleftrightarrow & \text{structures} & \longleftrightarrow & \text{of } \Gamma(J) \text{ with} \\ & & \text{on}\{1..n\} & & p^{-1}(\otimes) = \{1..n\} \end{array}$$

Lemma: $\Phi_{H,J}(n) = \mathbb{E}\left|\text{lifts of } \begin{array}{ccc} & & \widehat{\Gamma(J)} \\ & \nearrow & \big\downarrow P \\ \Gamma(H) & \longrightarrow & \Gamma(J) \end{array}\right|$

Open Problems

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- Same questions for other $Aut(\mathbf{F}_k)$-orbits of words (subgroups)

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- Understand completely $\mathbb{E}\left|\text{fix}\left(\alpha_{S_n}(w)\right)\right|$

Thank You!