On topological full group of minimal homeomorphisms of a Cantor set

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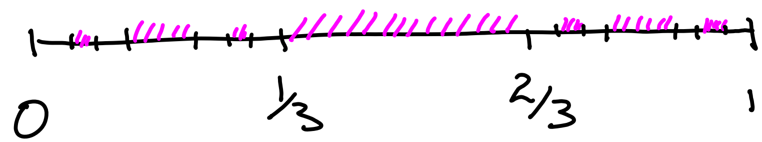
(I) A Cantor set



Th. A totally disconnected compact metric perfect space is homeomorphic to a Cantor set.

Different realizations:

(i) Space of sequences

A - finite alphabet, $\{0,1\}$ - binary alphabet

$$\Omega = A^{\mathbb{N}}, \quad \Omega \ni \omega = \omega_1 \omega_2 \ldots \omega_n \ldots, \quad \omega_n \in A$$

or $A^{\mathbb{Z}} \ni \eta = \ldots \eta_{-1} \eta_0 \eta_1 \ldots$, bi-infinite sequences

Tychonoff topology on Ω ($\Leftrightarrow$ topology of coordinatewise convergence).

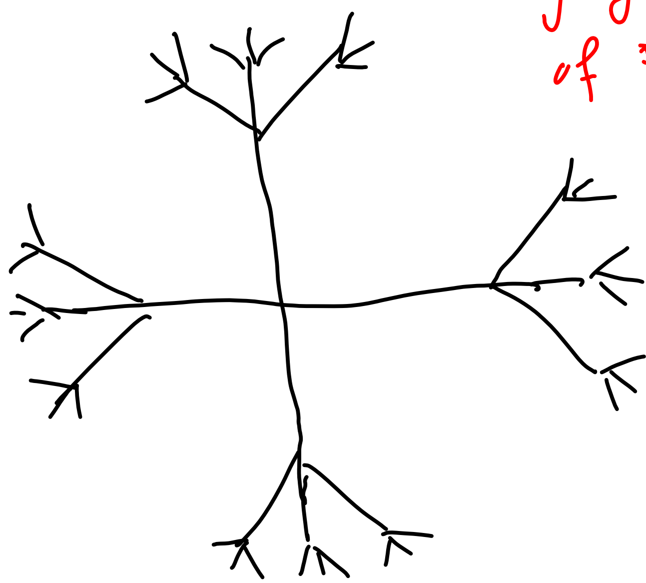
$\tau : \Omega \to \Omega$, $\quad (\tau(\omega))_n = \omega_{n+1}$ – shift

(Ω, τ) – full shift

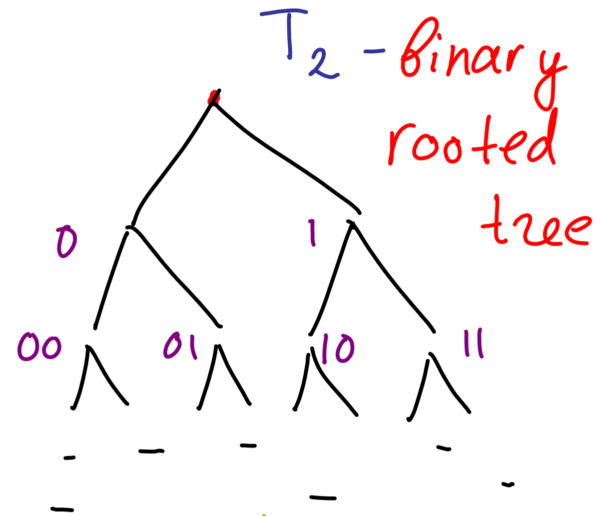
$(X, \tau|_X)$ – subshift $\quad X \subset \Omega$ – closed, τ-invariant subset

no isolated points in $X \Rightarrow (X, \tau)$ – Cantor system.

(ii) Boundary of a tree



Cayley graph of F_2



$\partial T = \{0,1\}^{\mathbb{N}}$

$$\begin{cases} \alpha(0w) = 1w \\ \alpha(1w) = 0\alpha(w) \end{cases} \qquad w \in \partial T$$

$\partial T \approx$ Cantor

$$\begin{cases} \alpha(0^n 1 w) = 1^n 0 w \\ \alpha(1^\infty) = 0^\infty \end{cases}$$

α - odometer

(iii) Bratelli diagrams, Vershik transformations

II Minimal Cantor Systems

$$(X, T)$$

Cantor set (X), homeomorphism (T)

Def. (X,T) is *minimal* if orbit

$$\mathcal{O}(x) = \{ T^n(x) : n \in \mathbb{Z} \}$$

of each point $x \in X$ is dense in X

($\iff$ no proper non empty T-invariant closed subsets)

Example. (i) $(\partial T_2, \text{odometer})$ – minimal system

(ii) $(\{0,1\}^{\mathbb{Z}}, \tau)$ – not a minimal system (a lot of periodic points).

Example. Morse system $\{0,1\}$ – alphabet

Blocks (words):

$$B_0 = 0, \qquad B_1 = B_0 \bar{B}_0$$

$$B_{n+1} = B_n \bar{B}_n ,$$

where $\bar{B}_n$ is the complement of B_n obtained by interchanging the 1's and 0's in B_n

$B_n \prec B_{n+1}$ $\qquad x^+ = \lim\limits_{n \to \infty} B_n$ – right infinite sequence

prefix

Prouhet, Tue, Morse

$$x^+ = 01101001100101101001011001101001 \ldots$$

$\{0,1\}^{\mathbb{Z}} \supset M = \{$ the set of sequences containing blocks that do appear in $x^+ \}$

(M, τ) – minimal system.

The same example via substitution:

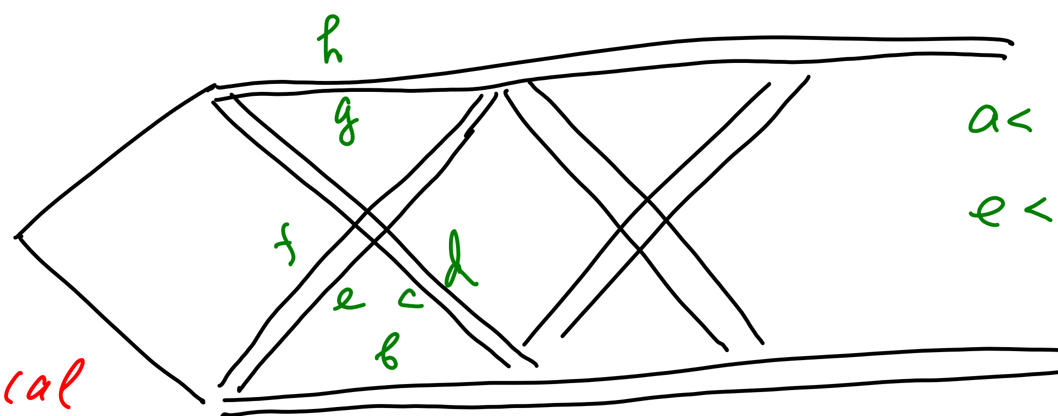
$$\sigma : \begin{cases} 0 \to 01 \\ 1 \to 10 \end{cases} \qquad B_n = \sigma^n(B_0)$$

$$x^+ = \lim_{n \to \infty} B_n$$

Example

$$\begin{cases} 0 \to 0011 \\ 1 \to 0101 \end{cases}$$

substitutional dynamical system



$a < b < c < d$

$e < g < f < h$

Bratelli diagram, Vershik map

(iii) Toeplitz shifts

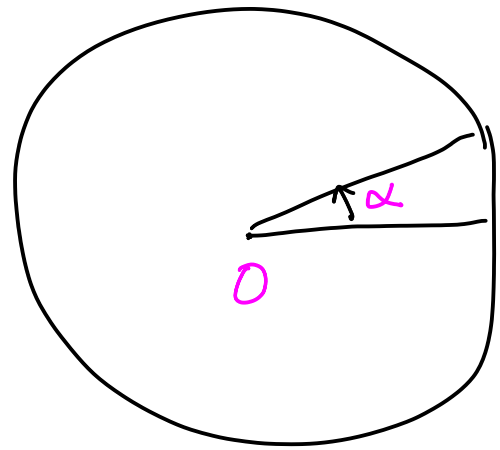
Def. A bi-infinite sequence $x \in A^{\mathbb{Z}}$ is a Toeplitz sequence if the set of integers can be decomposed into arithmetic progressions such that entry x_i is constant on each arithmetic progression.

$$A^{\mathbb{Z}} \ni x - \text{Toeplitz} \Rightarrow (X, \tau) - \text{minimal Cantor system}$$

$$X = \overline{O_\tau(x)} \quad - \text{closure of orbit}$$

(iv) Sturmian shifts

$$\text{'}\mathbb{T}\text{'} = [0,1) = \mathbb{R}/\mathbb{Z} = S^1$$



$$T_\alpha(x) = x + \alpha \pmod 1$$

(rotation by angle α, irrational)

$\mathcal{P}$ - partition: $[0,1) = [0,\alpha) \sqcup [\alpha,1)$

$[0,1) \ni t \longrightarrow$ itinerary $x^{(t)} = \left(x_i^{(t)}\right)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$

$$x_i^{(t)} = \begin{cases} 0 & \text{if } T_\alpha^i(t) \in [0,\alpha) \\ 1 & \text{if } T_\alpha^i(t) \in [\alpha,1) \end{cases}$$

$X = \overline{\{x^{(t)} : t \in [0,1)\}}$ - closure

(X, τ) - minimal Cantor

Topological entropy

(X, τ) - subshift of $(A^{\mathbb{Z}}, \tau)$

$$h(X,\tau) = \lim_{n\to\infty} \frac{1}{n} \log |B_n(X)|$$

$|B_n(X)|$ = # of n-blocks appearing in points of X

h is invariant of topological conjugacy

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "S"] & Y \end{tikzcd}$$

$f \circ T = S \circ f$, f - homeomorphism.

and of flip conjugacy: $T \sim S$ or $T \sim S^{-1}$.

III Topological Full Group (TFG)

(X, T) - Cantor system

Def. (i) Full group of (X,T) is $[T]$ - the subgroup of all homeomorphisms $S \in \mathrm{Homeo}\, X$ s.t. $\forall x \in X$

$$S(x) \in \mathcal{O}(x) = \{T^n(x) : n \in \mathbb{Z}\}$$

$$\Updownarrow$$

$$S(x) = T^{n_S(x)}(x)$$

$n_S : X \to \mathbb{Z}$ Borel measurable cocycle

(ii) $G_T = [[T]]$ – topological full group:

$$G_T = \{ S \in [T] : n_S(x) - \text{continious} \}$$

$S \in G_T \Leftrightarrow \exists$ a finite clopen partition $\{C_1, \dots, C_k\}$ of X and a set of integers $\{n_1, \dots, n_k\}$ s.t.

$$S|_{C_i} = T^{n_i}|_{C_i} \quad \forall i = 1, \dots, k$$

$[T]$ is "huge", $[[T]]$ is countable

Th. [Giordano, Putnam, Skau]. Let (X,T) and (Y,S) be Cantor minimal systems.

isomorphism

(i) They are orbit equivalent $\Leftrightarrow$ $[T] \simeq [S]$

(ii) They are $\underbrace{\text{flip conjugate}}$ $\Leftrightarrow$ $G_T \simeq G_S$ $\Leftrightarrow$ $G_T' \simeq G_S'$.

$T \sim S$ or $T \sim S^{-1}$

Th. [GPS, Bezugliy-Medynets, Matui]

1) G_T is indicable $(\exists\, \varphi : G_T \twoheadrightarrow \mathbb{Z})$

2) The commutator subgroup G_T' is simple and if $N \triangleleft G_T$ then $G_T' \leq N \leq G_T$.

3) G_T' is finitely generated $\Longleftrightarrow$ (X,T) is topologically isomorphic to a minimal subshift over a finite alphabet.

4) G_T' is not finitely presented.

5) There is a normal subgroup $I \triangleleft G_T$ with $G_T / I \simeq \mathbb{Z}$ and two locally finite subgroups $A, B \leq I$ s.t. $I = A \cdot B$.

6). Any finite group can be embedded into G_T.
Also G_T contains $\bigoplus_{\mathbb{N}} \mathbb{Z}$.

if (X,T) is not odometer then Lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ embeds into G_T'.

Conjecture [Gri-Medynets] G_T is amenable.

Th. [K. Juschenko, N. Monod]. The TFG G_T of any minimal Cantor system is amenable.

IV The result.

Def. [A. Stepin, A. Vershik, 70-th] A group G is LEF (locally embeddable into finite groups) if for every finite subset $F \subset G$ there is a finite group H and a map $\varphi: G \longrightarrow H$ s.t

(i) φ is injective on F

(ii) $\varphi(gh) = \varphi(g)\varphi(h) \quad \forall g, h \in F$

[in the case G is finitely generated this is equivalent to: G is a limit of a sequence of finite groups in the space of marked groups. 1984]

LEF and Amenable are "independent"

$$\text{LEF} \Rightarrow \text{sofic} \Leftarrow \text{Amenable}$$

Th. [Gri - Medynets] For any Cantor minimal system (X,T) the topological full group G_T is LEF.

Cor. There are uncountably many finitely generated simple LEF groups. [as $\forall\ h \geq 0$ there is a minimal Cantor subshift with topological entropy h].

V On the proof.

$A \subset X$ clopen $\qquad t_A : A \to \mathbb{N}$

$$t_A(x) = \min_k \{k \geq 1 : T^k(x) \in A\}$$

function of the first return

$A_k = \{x \in A : t_A(x) = k\}$, $\quad k \in K = \mathrm{Range}(t_A)$

$A_k, TA_k, \ldots, T^{k-1}A_k$ – disjoint

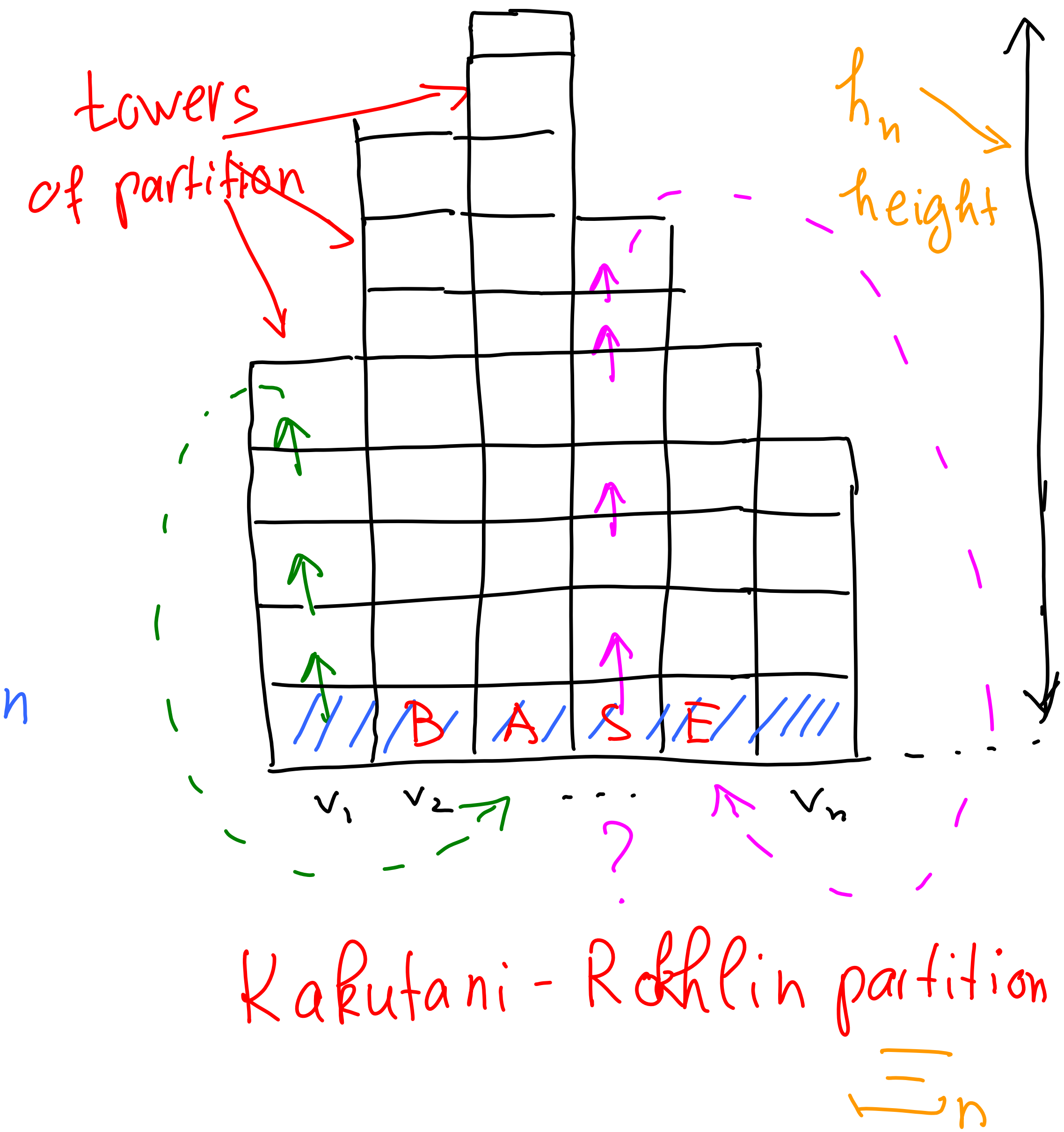
$$A = \bigsqcup_{k \in K} A_k$$

$$\boxed{X = \bigsqcup_{k \in K} \bigsqcup_{i=0}^{k-1} T^i A_k \quad \text{partition} \quad \Xi}$$

Fix $x_0 \in X$

sequence $\{\underset{\text{clopen}}{E_n}\}_{n=1}^{\infty}$

Ξ_n - clopen partition constructed by $\mathcal{L}_{E_n}$



Conditions:

(1) $\{\Xi_n\}_{n\geq 1}$ generate the topology of X.

(2) Ξ_{n+1} refines Ξ_n

(3) $\bigcap_n B(\Xi_n) = \{x_0\}$

$V_n = \{v_1, v_2, \dots, v_n\}$

$$\Xi_n = \{T^i B_v^{(n)} : 0 \le i \le h_v^{(n)} - 1,\ v \in V_n\}$$

Fix $\{m\}_{n \ge 1}$, $m_n \longrightarrow \infty$ as $n \to \infty$. Take a subsequence of $\{\Xi_n\}_{n \ge 1}$ so that additionally

(4) $h_n \ge 2m_n + 2$, where $h_n = \min_{v \in V_n} h_v^{(n)}$

(5) The sets $T^i B(\Xi_n)$ have the property

$$\operatorname{diam}(T^i B(\Xi_n)) < \frac{1}{n} \quad \text{for } -m_n \leq i \leq m_n$$

Remark. (1) - (4) do not need minimality of T (aperiodicity is enough). (5) holds only for minimal systems.

Def. Fix $n \geq 1$. $P \in G_T$ is n-permutation if

(i) its orbit cocycle $n_P(x)$ is compatible with the partition

(ii) $\forall x \in T^i B_v^{(n)}$ $\quad (0 \leq i \leq h_v^{(n)} - 1, \ v \in V_n)$

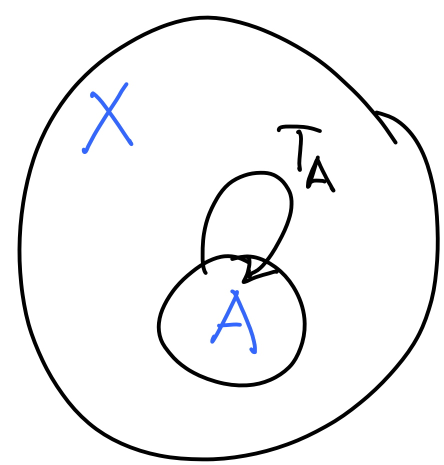
$$0 \leq n_P(x) + i \leq h_v^{(n)} - 1$$

[i.e. atoms of partition Ξ_n are permuted only within each tower]

Def.

$$T_A(x) = \begin{cases} T^{t_A(x)}(x) & \text{if } x \in A \\ x & \text{if } x \notin A \end{cases}$$

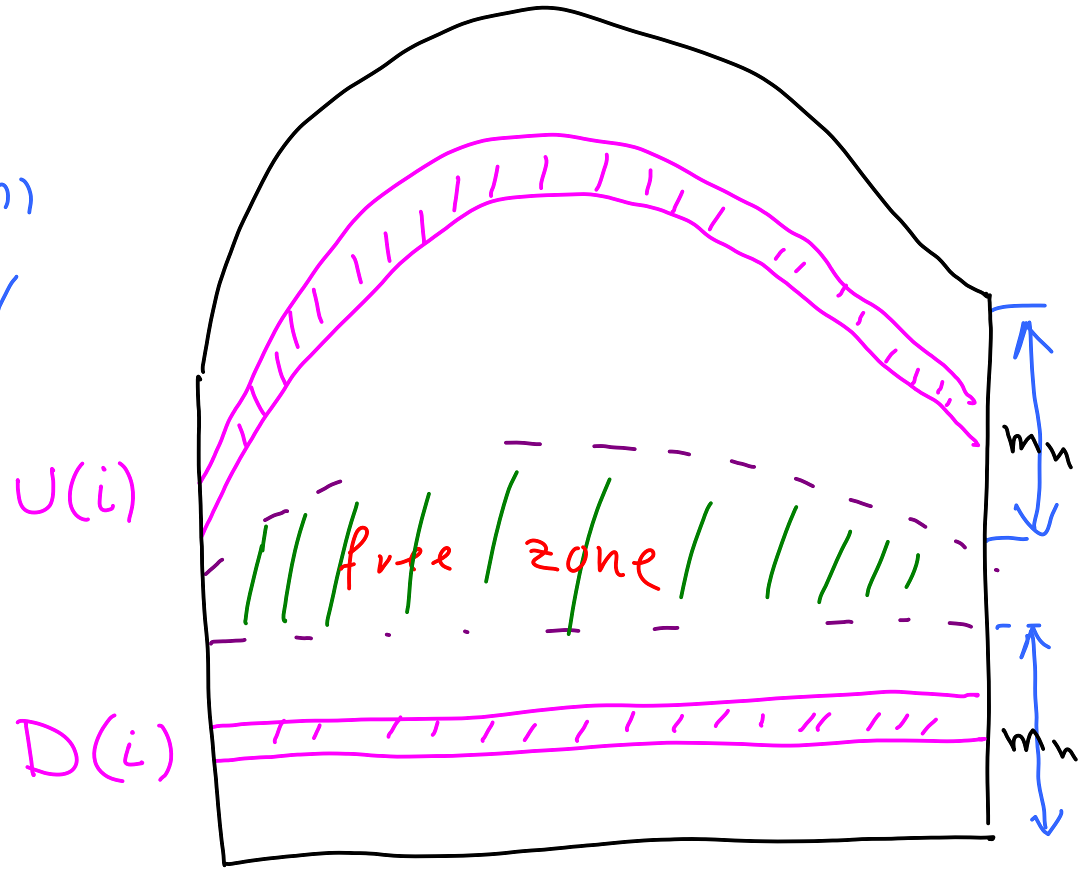
induced transformation



$$0 \leq i \leq m_n$$

$$U(i) = \bigsqcup_{v \in V_n} T^{h_v^{(n)} - i - 1} B_v^{(n)}$$

$$D(i) = \bigsqcup_{v \in V_n} T^{i} B_v^{(n)}$$



"strips" at distance i
from top and bottom respectively

Def. $R \in G_T$ is called an n-rotation with the rotation number $\leq r$ if there are subsets $S_U, S_D \subset \{0, 1, \ldots, m_n\}$ s.t.

$$R = \prod_{i \in S_U} \left(T_{U(i)}\right)^{\ell_i} \times \prod_{j \in S_D} \left(T_{D(j)}\right)^{k_j}$$

induced maps

$|\ell_i| \leq r$, $|k_j| \leq r$

$$\mathrm{rot}(R_1 R_2) \leq \mathrm{rot}(R_1) + \mathrm{rot}(R_2)$$

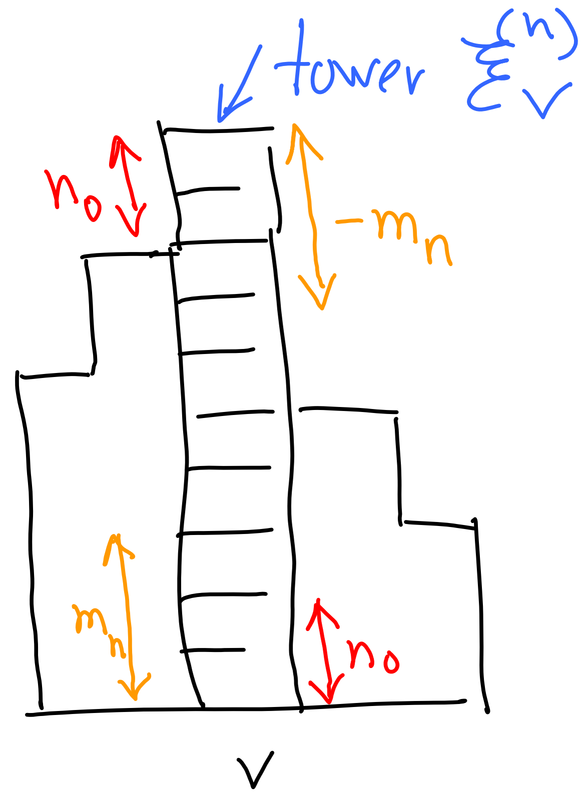
Proposition. Let $Q \in G_T$. Then there exists $n_0 > 0$ s.t. for all $n \geq n_0$, the homeomorphism Q can be represented as $Q = PR$, where P is an n-permutation and R is an n-rotation with rotation number not exceeding $\underline{1}$.

Furthermore, the permutation P can be represented (in a unique way) as a product of permutations $P_1, \ldots, P_{r_n}$ meeting the following conditions:

(i) The permutation P_v acts only within T-tower $\xi_v^{(n)}$

(ii) $P_v(i) = P_w(i)$ for all

$i \in [-m_n, m_n]$, $v, w \in V_n$



(iii) The map P_v induces a permutation of $\{0, 1, \ldots, h_v^{(n)} - 1\}$ with the property that

$$\forall i \qquad d_v^{(n)}(P_v(i), i) \leq n_0$$

(iv) The rotation R acts only on levels which are within the distance n_0 to the top or the bottom of the partition

(v) $Q = P_1 R_1 = P_2 R_2 \Rightarrow P_1 = P_2, \quad R_1 = R_2$

uniqueness of decomposition.

(vi) For any finite subset $\{Q_1, \dots, Q_k\}$ of G_T there is $n_1 > n_0$ s.t in the decomposition $Q_i = P_i R_i$ all permutations P_i are different.

Proof of theorem. $F \subset G_T$, $|F| < \infty$

Find $n \in \mathbb{N}$ s.t $\forall Q \in F^2$, $Q = P_Q R_Q$

$P_Q \neq P_Z$ for $Q, Z \in F^2$, $Q \neq Z$

$\exists\, d \in \mathbb{N}$ s.t. all n-rotations R_Q, $Q \in F$ are supported by levels $[-d, d]$, $n \gg 1$

Can choose $n \in \mathbb{N}$ s.t. $\forall\, Q \in F$

$$S_{Q,v}^{\pm 1}(i) = \tilde{S}_{Q,w}^{\pm 1}(i) \qquad \forall\, i \in [-d, d] \quad \forall\, v, w \in V_n$$

$$\tilde{S}_Q = \prod_{v \in V_n} \tilde{S}_{Q,v}$$

$\Rightarrow$ $\tilde{S}_Z^{-1} R_Q \tilde{S}_Z$ is an n-rotation, $\forall\, Z, Q \in F$

H = group of all n-permutations

Define $\varphi : F^2 \longrightarrow H$

$$\varphi(Q) = \varphi(P_Q R_Q) := P_Q$$

$$\varphi(QZ) = \varphi(P_Q R_Q P_Z R_Z)$$

$$= \varphi\big((\underbrace{P_Q P_Z}_{n\text{-permutation}})\, \underbrace{\underbrace{P_Z^{-1} R_Q P_Z}_{} R_Z}_{n\text{-rotations}}\big)$$

$$\Longrightarrow \varphi(QZ) = P_Q P_Z = \varphi(Q)\varphi(Z) \qquad \blacksquare$$

Cor. G_T' is not finitely presented.