Stackable groups

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Joint work with

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A "wish list" from geometric group theory

Find a geometric/algorithmic property that:

1) Makes computing the word problem for **3-manifold** groups tractable.

Eg: Asynchronously "automatic" over an indexed language *(Bridson, Gilman)*, *but* not regular/automatic *(ECHLPT, N Brady)*.

2) Uniformly captures the algorithms for building van Kampen diagrams arising from both finite convergent **rewriting systems** and **almost convexity**.

3) Facilitates computation of **asymptotic** "filling" **invariants** (from van Kampen diagrams, e.g. Dehn, isodiametric functions).

4) Captures the complexity/ simplicity of the word problem for **Thompson's group** F.

GGT theme: Measure algorithmic properties using normal forms + conditions on Γ; e.g. automatic, combable ("fellow travel").

Stackable groups I: Definition

Group $G = \langle A \rangle$, $|A| < \infty$, $A = A^{-1}$, Γ = Cayley graph

Notn. Normal forms $\mathcal{N} = \{y_g \mid g \in G\} \subset A^*$; $\mathcal{N} \hookrightarrow\!\!\!\rightarrow G$ $\quad (y_\epsilon = 1)$

$\vec{E}(\Gamma) := \{e_{g,a} := g \bullet\!\!\xrightarrow{a}\!\!\bullet ga \mid g \in G, a \in A\}$ = directed edges of Γ

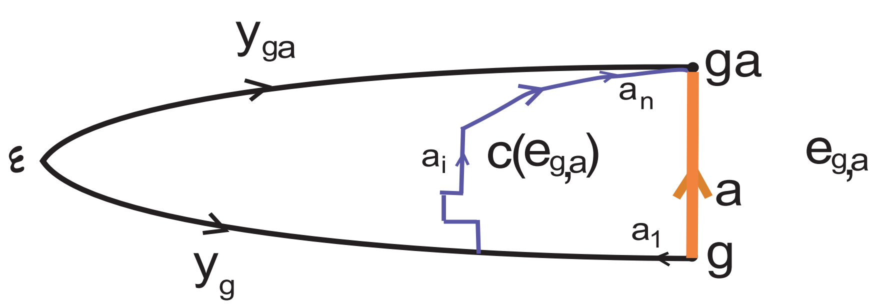
$\vec{E}_d := \{e_{g,a} \mid y_g\ a = y_{ga}$ or $y_g = y_{ga}\ a^{-1}\}$ = **degenerate** edges

$\vec{E}_r := \vec{E}(\Gamma) \setminus \vec{E}_d$ = **recursive** edges

Defn. G is **stackable** (over A) if $\exists$

- normal forms $\mathcal{N}$, • constant k, and
- well-founded strict partial ordering $<$ on $\vec{E}_r$, such that

$\forall\ e_{g,a} \in \vec{E}_r$: $\exists$ directed path labeled $a_1 \cdots a_n$ from g to ga in Γ
satisfying: $n \leq k$ and
$\forall$ directed edges $e_i := e_{ga_1 \cdots a_{i-1}, a_i}$ in the path:
either $e_i \in \vec{E}_r$ and $e_i < e_{g,a}$, or $e_i \in \vec{E}_d$.



Stacking map $c : \vec{E}_r \to A^*$:
Choice of $c(e_{g,a}) = a_1 \cdots a_n$.

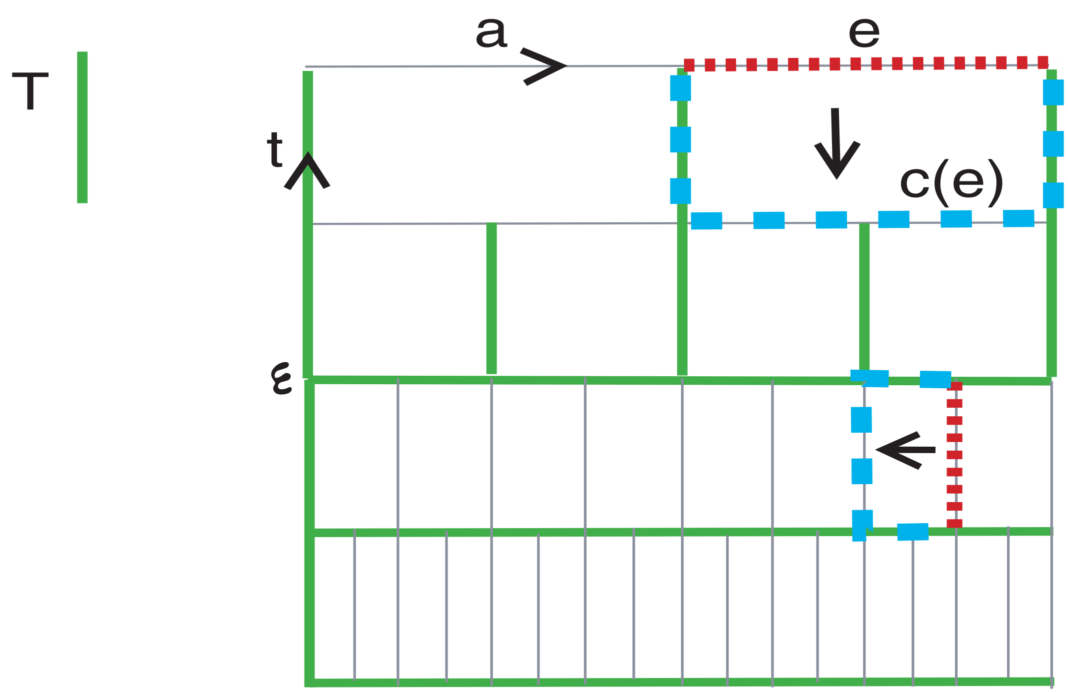
Note $c(e_{g,a}) \neq a$.

Stackable groups, II: Picture

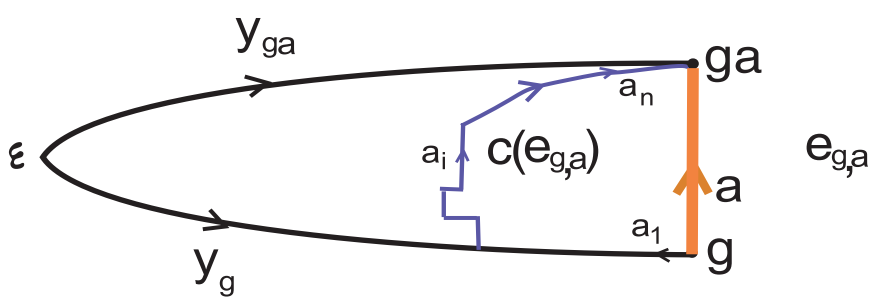
Prop. (B,H) *Stackable* $\Rightarrow$ *finite presentation,* $\mathcal{N}$ *prefix-closed.*

Rmks. • $\mathcal{N}, \vec{E}_d \quad \longleftrightarrow \quad$ tree $T \subseteq \Gamma$.
• $y_g \in \mathcal{N} \quad \longleftrightarrow \quad$ simple path in Γ.
• $<$ on $\vec{E}_r \quad \longleftrightarrow \quad$ "flow" of non-tree edges toward identity.

Ex. $BS(1,2) = \langle tat^{-1} = a^2 \rangle$
$\mathcal{N} = \{t^{-i}a^j t^k \mid i,k \in \mathbb{N}_0,\ j \in \mathbb{Z},$ and either $0 \in \{i,k\}$ or $2 \not| \ j\}$



Stackable groups, III: Algorithms



Extend to $c' : \vec{E}(\Gamma) \to A^*$:

$$c'(e_{g,a}) := \begin{cases} c(e_{g,a}) & \text{if } e_{g,a} \in \vec{E}_r \\ a & \text{if } e_{g,a} \in \vec{E}_d. \end{cases}$$

Defn. G is **algorithmically stackable** if
graph$(c') := \{(y, a, c'(e_{y,a})) \mid y \in \mathcal{N}, a \in A\} \subset A^* \times A \times A^*$
is *computable*, and
automatically stackable, or **autostackable**, if *graph*(c') (viewed as padded strings over $(A \cup \{\$\})^3$) is a *regular language*.

Prop. (B,H) *Algorithmically stackable* $\Rightarrow$ *solvable word problem.*

Stacking reduction algorithm on words in A^*:

(1) If $y \in \mathcal{N}$, $a \in A$, $ya \notin \mathcal{N}$, $z \in A^*$, reduce $y\ a\ z \to y\ c(e_{y,a})\ z$.

(2) Free reduction

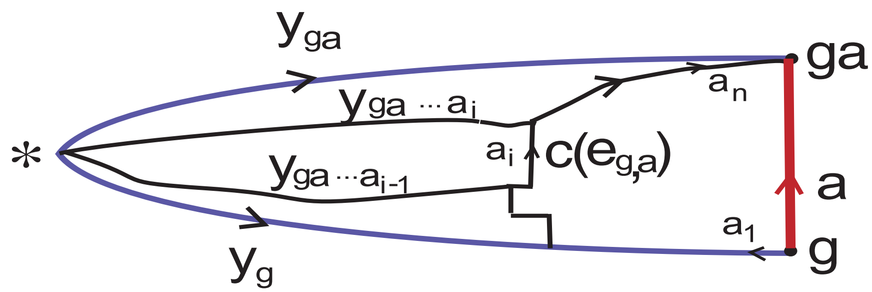
Repetition of (1-2) reduces any $w \in A^*$ to normal form.

Inductive procedure for constructing van Kampen diagrams

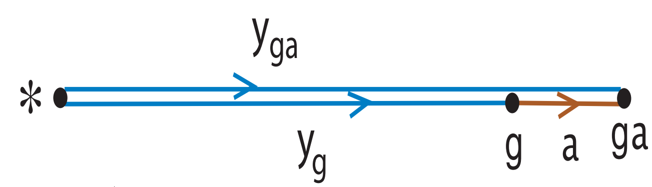
Stackable gp G, $\mathcal{P} = \langle A \mid R \rangle$ stacking pres., $\mathcal{N} = \{y_g \mid g \in G\}$

Defn. $w \in A^*$ with $w =_G \epsilon \Rightarrow \exists$ **van Kampen diagram** Δ:
(i) finite, planar, contractible, combinatorial 2-complex,
(ii) $\partial\Delta$ = path at basepoint $*$ labeled w, ∂(2-cells) $\in R$.

Step I: Noetherian induction using well-founded strict partial ordering $<$ on $\vec{E}_r \Rightarrow$ algorithm to fill "icicles" = v. K. diagrams for words of the form $w = y_g\ a\ y_{ga}^{-1}$:



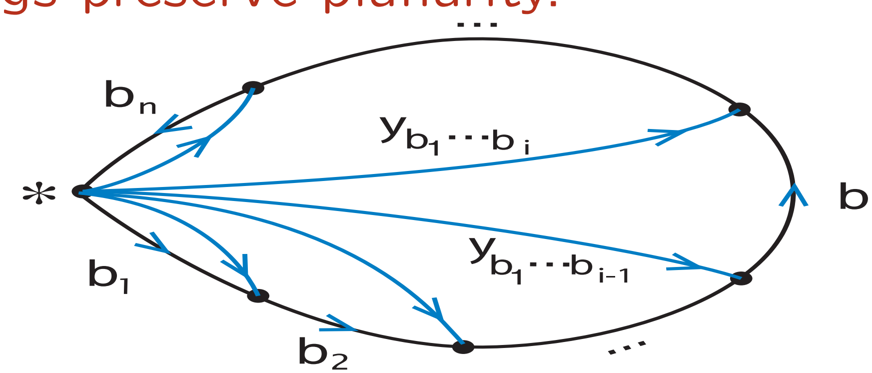
$e_{g,a}$ recursive



$e_{g,a}$ degenerate

Rmk. y_g paths simple in $\Gamma \Rightarrow$ gluings preserve planarity.

Step II: Build a van Kampen diagram for any $w = b_1 \cdots b_n \in A^*$ with $w =_G \epsilon$ using "seashell":
(Wish 3)



Autostackable examples: Finite convergent rewriting systems

Def. A finite **convergent rewriting system** (**CRS**) for a group G consists of a finite generating set $A = A^{-1}$ and a finite set of defining relations $R = \{u \to v\}$ with $u, v \in A^*$ such that the rewritings $xuy \to xvy$ for all $x, y \in A^*$ satisfy:

- $\mathcal{N} := A^* - \cup_{u \to v \in R} A^* u A^*$ is a set of normal forms for G.
- $x > y$ iff $x \to \cdots \to y$ is a well-founded strict partial ordering.

Ex: Finite, free, abelian, surface (Le Chenadec), prime alt. / torus knot (Chouraqui/Benninghofen, Kemmerich, Richter), π_1 *closed 3-manifold non-hyp. uniform geometry* (H, Shapiro), *iterated Baumslag-Solitar* (Gersten) groups;
closed under graph/semidirect product and certain amalgamated products, HNN extensions (H, Meier/Groves, Smith)

Thm. (Brittenham, H.) *Finite CRS* $\Rightarrow$ *autostackable.*

Stacking: $\mathcal{N}$: Irreducible words (above)

$<$: Define $prl(e_{g,a})$:= number of (shortest prefix) rewritings from $y_g\ a$ to y_{ga}, and $e' < e$ iff $prl(e') < prl(e)$.

c: Write $y_g\ a = \tilde{y}\tilde{u}a$ where $\tilde{u}a \to v \in R$; then $c(e_{g,a}) := \tilde{u}^{-1}v$.

Autostackable examples: Iterated Baumslag-Solitar groups

$$G_k = \langle a_0, a_1, ..., a_k \mid a_i^{a_{i+1}} = a_i^2; 0 \le i \le k-1 \rangle$$

G_k is autostackable: G_k has a finite CRS (Gersten).
But the minimal isoperimetric (Dehn) function for G_{k+1} grows at least as fast as a k-fold iterated tower of exponentials

$n \mapsto \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}_{\text{k times}}$. (Gersten)

Rmk. Asymptotic properties of autostackable groups are *much* less restrictive than those of automatic groups.

Automatic groups

Defn. $G = \langle A \rangle$ is **automatic** if $\exists$ • constant k and
• regular language $\mathcal{N} = \{y_g \mid g \in G\}$ of normal forms such that
$\forall\ g \in G,\ a \in A$: the paths y_g, y_{ga} (synchronously) k-fellow travel.

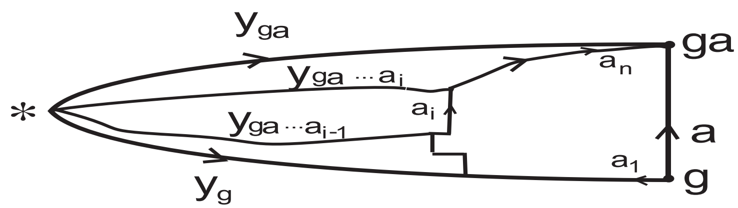
Comparing autostackable and automatic structures

Thm. *(Brittenham, H.) All shortlex automatic groups (which include all word hyperbolic groups) are autostackable.*

Autostackable

$\{(y, a, c'(e_{y,a})) \mid y \in \mathcal{N}, a \in A\}$ is regular.

v.K. d. icicle filled inductively:

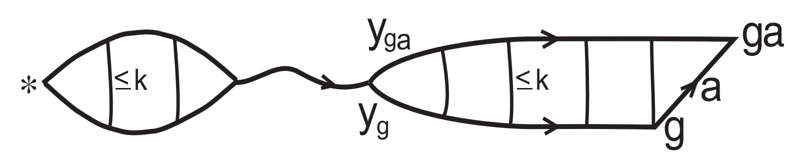


Dehn fcn. may be iterated exponential.

Automatic

$\{(y, a, y_{ya}) \mid y \in \mathcal{N}, a \in A\}$ is regular.

v.K. d. icicle is k-thin:



Dehn fcn. $\preceq$ quadratic.

For M = 3-manifold with any of the 8 uniform geometries:

Thm. *(B,H)* $\pi_1(M)$ *is autostackable.*

(Wish (1))

Thm. *(ECHLPT)* $\pi_1(M)$ *is automatic iff geometry is not Nil, Sol.*

Stackable examples: Almost convex groups

Group $G = \langle A \rangle$, $|A| < \infty$, $A = A^{-1}$, Γ = Cayley graph

Notn. For edge $e_{g,a}$ define $\tilde{d}(e_{g,a}) := \min\{d_\Gamma(\epsilon, g), d_\Gamma(\epsilon, ga)\}$.

Defn. (Cannon) G is **almost convex** (over A) if $\exists$

- constant k such that

$\forall r \in \mathbb{N}$, $\forall\ g, h \in G$ with $d_\Gamma(\epsilon, g) = r = d_\Gamma(\epsilon, h)$ and $d_\Gamma(g, h) \leq 2$:
$\exists$ directed path labeled $a_1 \cdots a_n$ from g to h in Γ satisfying:

$n \leq k$ and

$\forall$ directed edges $e_i := e_{ga_1 \cdots a_{i-1}, a_i}$ in the path: $\tilde{d}(e_i) < r$.

Thm. (Brittenham, H.) *Almost convex* $\Rightarrow$ *algorithmically stackable.*

Stacking: $\mathcal{N} :=$ {shortlex normal forms}.

$<$: Define $e' < e$ iff $\tilde{d}(e') < \tilde{d}(e)$.

c: $c(e_{ga}) :=$ almost convexity path.

(Wish (2))

Stackable examples: Thompson's group F

$F < PL_0([0,1])$; slopes 2^i, breakpoints in $\mathbb{Z}_2$.

$F = \langle x_0, x_1 \mid [x_0x_1^{-1}, x_0^{-1}x_1x_0], [x_0x_1^{-1}, x_0^{-2}x_1x_0^2]\rangle$.

Rmk. The word problem is solvable for F. How solvable is it?

F is not (minimally) almost convex (Cleary, Taback; Belk, Bux).
Open Q's. (Guba) Is F automatic? Does F have a finite convergent rewriting system? (Wish (4))

Thm. (Cleary, H., Stein, Taback) *Thompson's group F is (algorithmically) stackable.*

Rmk. The stacking normal forms are **context free**:
$\mathcal{N}$:= set of all $w \in A^*$ such that

(1) $w \in A^* - \cup_{u \in U} A^*uA^*$ with $U := \{x_0^\eta x_0^{-\eta}, x_1^\eta x_1^{-\eta}, x_0^2 x_1^\eta \mid \eta = \pm 1\}$,

(2) For each prefix v of w, $\text{expsum}_{x_0}(v) \leq 0$.

Prop. (Cleary, H., Stein, Taback) *$\mathcal{N}$ is a (6,0)-quasigeodesic set of normal forms for F.*

Recall: A “wish list” from geometric group theory

Find a geometric/algorithmic property that:

1) Makes computing the word problem for **3-manifold** groups tractable.

2) Uniformly captures the algorithms for building van Kampen diagrams arising from both finite convergent **rewriting systems** and **almost convexity**.

3) Facilitates computation of **asymptotic** “filling” **invariants** (from van Kampen diagrams, e.g. Dehn, isodiametric functions).

4) Captures the complexity/ simplicity of the word problem for **Thompson’s group** F.

Filling functions, I: Diameter

$G = gp(\mathcal{P})$, $\mathcal{P} = \langle A \mid R \rangle$, $A = A^{-1}$, X = Cayley 2-cx

$w \in A^*$ with $w =_G \epsilon \Rightarrow \exists$ planar van Kampen diagram Δ: $\partial\Delta = w$

Notn. • $\pi_\Delta : \Delta \to X$, $\pi_\Delta(*) = \epsilon$

• d_Y = path metric in 1-skeleton of 2-cx Y. ($Y = \Delta$ or X.)

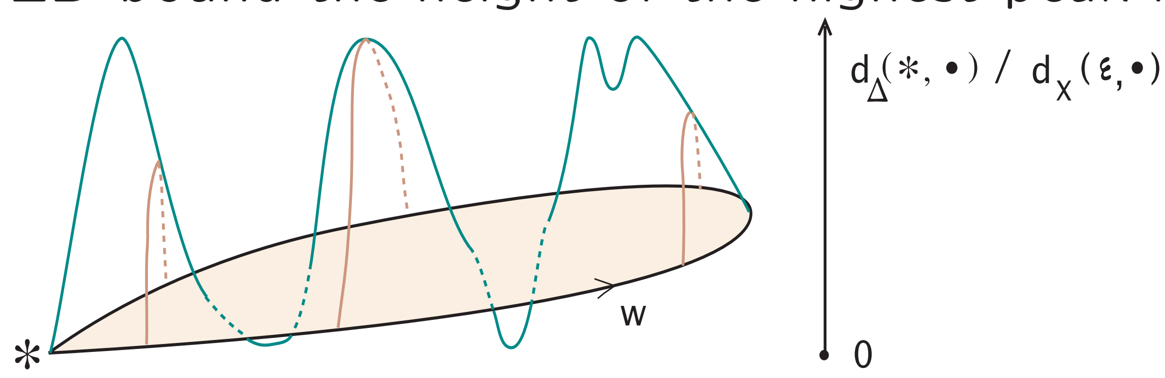
Defn. $(G, \mathcal{P})$ admits an **intrinsic** **[extrinsic]** **diametric inequality** for nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ if

$\forall\ w \in A^*$ with $w =_G \epsilon$, $\exists$ a v.K. diagram Δ for w such that

$\forall v \in \Delta^0$, $d_\Delta(*, v) \leq f(l(w))$ $[d_X(\epsilon, \pi_\Delta(v)) \leq f(l(w))]$.

Rmk. $\Delta \subseteq \mathbb{R}^2$; embed $\Delta^{(1)} \hookrightarrow \mathbb{R}^3$, $p \mapsto (p, d_\Delta(*, p))$ or $(p, d_X(\epsilon, \pi_\Delta(p)))$.

Picture: ID / ED bound the height of the highest peak in Δ:

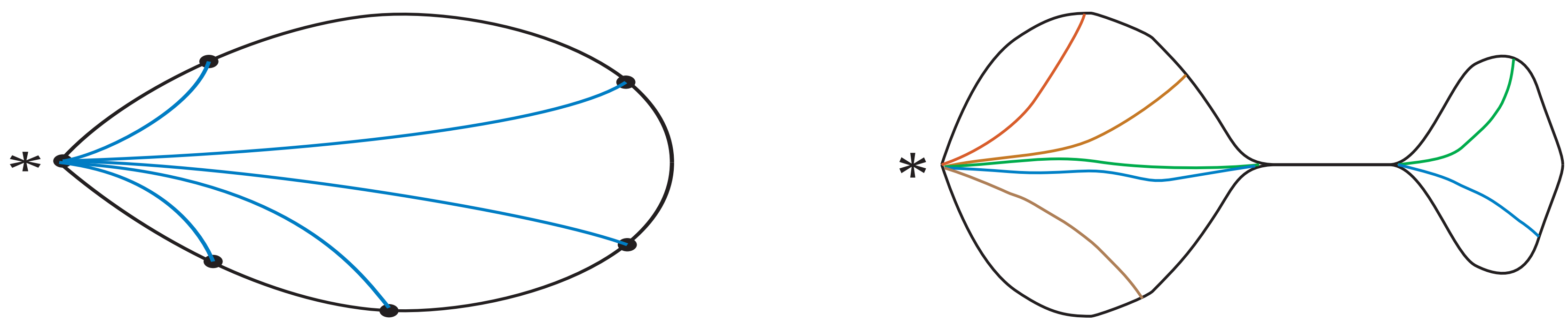


Filling functions, II: Combing van Kampen diagrams

Idea: Refine ID/ED: Measure "tameness" of peaks and valleys.

Def. A **van Kampen homotopy** of a v.K. diagram Δ is a continuous function $\Psi : \partial\Delta \times [0,1] \to \Delta$ such that

- $\forall p \in \partial\Delta$, $\Psi(p,0) = *$ and $\Psi(p,1) = p$,
- $\forall t \in [0,1]$, $\Psi(*,t) = *$, and
- $\forall p \in \partial\Delta^0$, $\Psi(p,t) \in \Delta^1$ for all $t \in [0,1]$.



Defn. $\tilde{d}_Y(y,\cdot)$ = **coarse distance** to vertex y in a 2-cx Y:

p = vertex: $\tilde{d}_Y(y,p) := d_Y(y,p)$

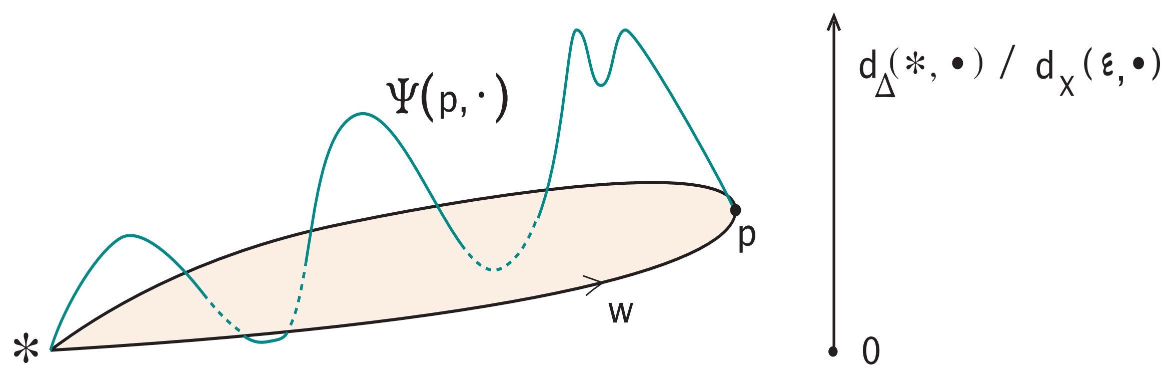
$p \in Int(\text{edge } e)$: $\tilde{d}_Y(y,p) := \min\{\tilde{d}_Y(y,v)) \mid v \in \partial(e)\} + \frac{1}{2}$

$p \in Int(\text{2-cell } \sigma)$: $\tilde{d}_Y(y,p) := \max\{\tilde{d}_Y(y,e) \mid e \in \partial(\sigma)\} - \frac{1}{4}$

Filling functions, III: Tame filling inequalities

Defn. $(G, \mathcal{P})$ admits an **intrinsic [extrinsic] tame filling inequality** for nondecreasing function $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ if
$\forall\ w \in A^*$ with $w =_G \epsilon$, $\exists$ a v.K. diagram Δ for w and
a v.K. homotopy $\Psi : \partial\Delta \times [0,1] \to \Delta$ such that
$\forall p \in \partial$ and $\forall 0 \leq s < t \leq 1$, $\quad \tilde{d}_\Delta(*, \Psi(p,s)) \leq f(\tilde{d}_\Delta(*, \Psi(p,t)))$
$[d_X(\epsilon, \pi_\Delta(\Psi(p,s))) \leq f(d_X(\epsilon, \pi_\Delta(\Psi(p,s))))]$.

Verbally: IT / ET bound: After a homotopy path reaches height $f(n)$, the path cannot return down to height n.



Rmk/Conj. (Tschantz) For some presentations, a finite function f may not exist.

Quasi-isometry invariants

Thm. (Gersten; Bridson, Riley; Brittenham, H) *The following are quasi-isometry invariants of groups, up to Lipschitz equivalence of nondecreasing functions:*

- *Intrinsic diameter inequality for f,*
- *Extrinsic diameter inequality for f,*
- *Extrinsic tame filling inequality for f, and*
- *Intrinsic tame filling inequality for f w.r.t. presentation with sufficiently large set of defining relations.*

Notn. G is in ET_f if G admits an extrinsic tame filling inequality for a function Lipschitz equiv. to f. Similarly ID_f, ED_f, IT_f.

$$\begin{array}{ccccc} & \mathrm{IT}_f & & \mathrm{ET}_f & \\ (2) & \Downarrow & & \Downarrow & (2) \\ & \mathrm{ID}_{\hat{f}} & \overset{\Rightarrow}{\not\Leftarrow} & \mathrm{ED}_{\hat{f}} & \\ & & (1) & & \end{array}$$

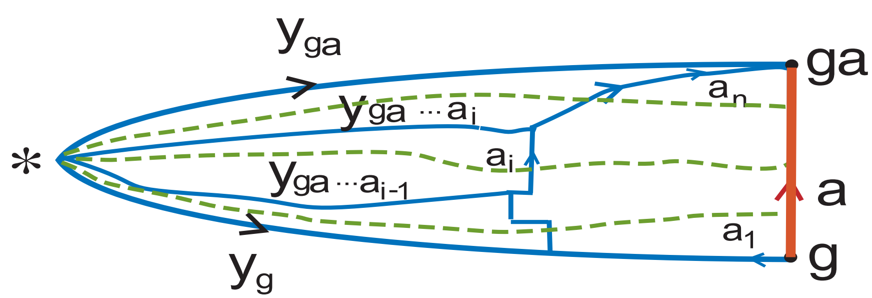
(1) Bridson-Riley
(2) Brittenham-H

$$\hat{f}(n) = \lceil f(n) \rceil$$

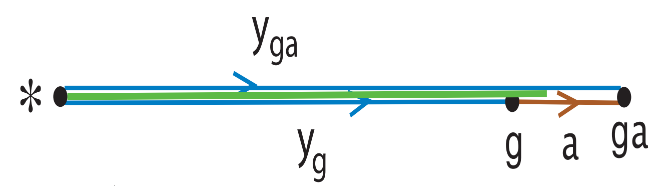
Inductive procedure for constructing van Kampen homotopies

Stackable gp G, $\mathcal{P} = \langle A \mid R \rangle$ stacking pres., $\mathcal{N} = \{y_g \mid g \in G\}$

Step I. Noetherian induction $\Rightarrow$ algorithm to build "edge fillings" of icicles = v. K. diagrams for words $y_g\ a\ y_{ga}^{-1}$:

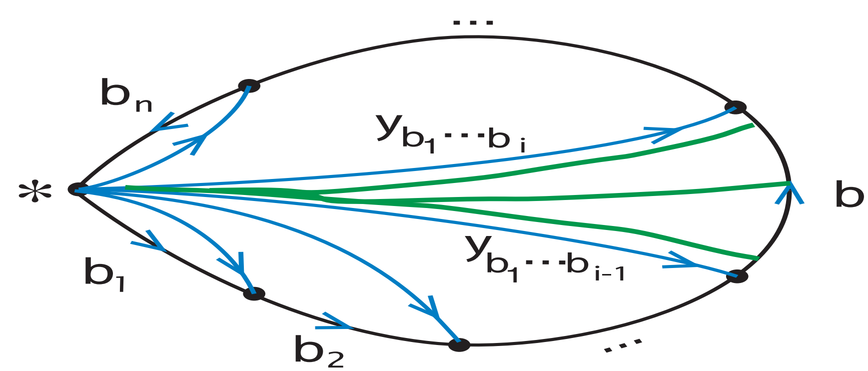


$e_{g,a}$ recursive



$e_{g,a}$ degenerate

Step II: Build a van Kampen homotopy for any $w = b_1 \cdots b_n \in A^*$ with $w =_G \epsilon$ using "seashell":



Rmk. Procedure enables algorithm to find bounds for tame filling inequalities. (Wish 3)

Tame filling invariants for stackable groups, I

Thm 1. (Brittenham, H.) *Stackable groups admit finite-valued intrinsic and extrinsic tame filling inequalities.*

Cor. (B, H) *Stackable groups admit tame combings (a la Mihalik-Tschantz).*

Conj. (Tschantz) ∃ fin. pres. non-tame-combable group.

Conj. ∃ finitely presented group that is not stackable with respect to any finite generating set.

Thm 2. (B, H) *Algorithmically stackable groups satisfy recursive intrinsic and extrinsic tame filling inequalities.*

Tame filling invariants for stackable groups, II

Thm 3. (B, H) *If G admits a finite convergent rewriting system, then G admits intrinsic and extrinsic tame filling inequalities for a function equivalent to the* **string growth complexity** *function $\gamma : \mathbb{N} \to \mathbb{N}$ defined by*

$$\gamma(n) := \max\{l(x) \mid w \to^* x,\ l(w) \leq n\}.$$

Thm 4. (B, H) *The following are equivalent for a group G:*

- *G is almost convex.*
- *G satisfies an intrinsic tame filling inequality for $id : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$.*
- *G satisfies an extrinsic tame filling inequality w.r.t. id.*

Thm 5. (Cleary, H., Stein, Taback; B, H) *Thompson's group F admits linear extrinsic and intrinsic tame filling inequalities.*

Rmk. Refines: F has linear intrinsic diameter inequality. (Guba)

Open Questions

(1) Cayley automatic groups (Kharlampovich, Khoussainov, Miasnikov) have a regular language of "normal forms" (not necessarily over a generating set) that fellow travel. Are all Cayley automatic groups autostackable, or vice-versa?

(2a) Is π_1 of every closed 3-manifold autostackable?

(2b) Determine closure properties for (auto)stackable groups.

Thm (Johnson) *The class of (algorithmically) stackable groups is closed under graph and semidirect products (and hence free/direct products).*

(2b') Is the fundamental group of a graph of autostackable groups, with edge groups "nice" (eg 1, $\mathbb{Z}$ or $\mathbb{Z}^2$), autostackable?

Open Questions, II

(3) Is there a global upper bound (eg iterated exponential, primitive recursive) on the Dehn fcn, or solution time for the word problem, for all autostackable groups?

(4) (Gilman) Is there an algorithm which, given a finite presentation $G = \langle A \mid R \rangle$ together with a division ordering on A^* and a well-founded strict partial ordering on (directed) edges of Γ, will halt and output an autostackable structure for G over A with respect to these orderings if one exists?