Artin groups of large type

Sarah Rees,

University of Newcastle, UK

Stevens Group Theory Webinar, October 6th 2011

Abstract: Artin groups of large type

Artin groups are those groups with presentations of the form

$$\langle a_1, \ldots, a_n \mid \overbrace{a_i a_j a_i \cdots}^{\mathsf{m_{ij}}} = \overbrace{a_j a_i a_j \cdots}^{\mathsf{m_{ij}}}, \quad i < j, \mathsf{m_{ij}} \in \mathbb{N} \cup \infty, \mathsf{m_{ij}} \geq 2 \rangle.$$

Braid groups, free groups, and free abelian groups provide examples, and Coxeter groups are natural quotients. Following Appel&Schupp, we say that such a group has large type if each $\mathsf{m_{ij}}$ is at least 3. It was proved 20 years ago by Appel and Schupp using small cancellation techniques that Artin groups of large type have soluble word and conjugacy problem; later Peifer and Brady&McCammond used small cancellation to prove many of these biautomatic.

I'll describe joint work with Derek Holt that describes the set L of shortlex minimal geodesics for any Artin group G of large type, given any ordering

of its (natural) generating set, proves both L and the set of all geodesics to be regular sets, and proves that L provides a shortlex automatic structure for G. Hence G admits a quadratic time solution to the word problem, using a reduction process that has parallels with Tits' solution to the word problem for Coxeter groups, and has a quadratic Dehn function. Subgroups generated by subgroups of the natural generating set (proved already to be Artin groups (see Appel&Schupp; Appel; Lek) are now seen to be quasiconvex (in fact convex) in G. This work builds on work of Mairesse and Mathéus that identifies the geodesics in dihedral (i.e. 2-generator) Artin groups.

Notation

$G = \langle X \rangle$ is a group, $|X| < \infty$.

A word over X is a string over $X^{\pm} := X \cup X^{-1}$, i.e. an element of $X^{\pm *}$. We use $=$ for equality of strings, $=_G$ for equality of group elements, $|w|$ for the string length of w, $|w|_G$ for $\min\{|u| : u =_G w\}$.

The shortlex word order puts u before v ($u <_{\text{slex}} v$) if either $|u| < |v|$ or $|u| = |v|$ but u precedes v lexicographically.

So

$$\textbf{cat} <_{\text{slex}} \textbf{dog} <_{\text{slex}} \textbf{cats}.$$

Artin groups

An Artin group is defined in terms of a Coxeter matrix, i.e. a symmetric matrix $(\mathsf{m_{ij}})$ with entries in $\mathbb{N} \cup \{\infty\}$, and off-diagonal entries all at least 2. It has a presentation

$$\langle a_1, \ldots, a_n \mid {}_{\mathsf{m_{ij}}}(a_i, a_j) = {}_{\mathsf{m_{ij}}}(a_j, a_i) \quad \text{for each} \quad i \neq j \rangle,$$

where we write ${}_m(a, b)$ for the alternating product $aba \cdots$ of length m, (later we'll also write $(a, b)_m$ for $\cdots bab$).

Adding the relations $a_i^2 = 1$ defines the associated Coxeter group.

An Artin group is said to be of **spherical** or finite type if the Coxeter group is finite, and of **dihedral** type if it is 2-generated.

It is of **large** or **extra-large** type if all $\mathsf{m_{ij}}$ are at least 3, or at least 4.

It is **right angled** if all $\mathsf{m_{ij}}$ are 2 or infinite.

The word and conjugacy problems

Let $G = \langle X \mid R \rangle$ be a group, $|X| < \infty$.

G has **soluble word problem** if $\exists$ algorithm to decide, given any input word w over X, '$w =_G 1$?'

G has **soluble conjugacy problem** if $\exists$ algorithm to decide, given any pair of input words w, v over X, '$\exists g \in G : gwg^{-1} =_G v$?'.

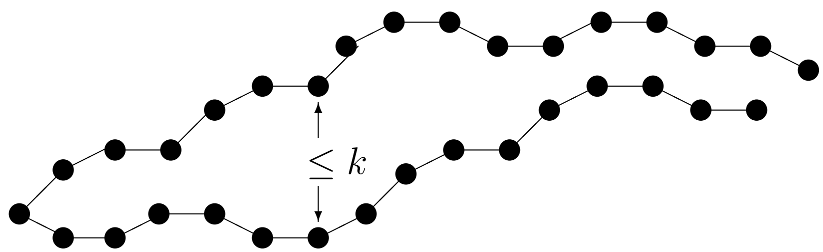
It's well known that both problems are theoretically insoluble even for finitely presented groups, and that there are groups with soluble word problem but insoluble conjugacy problem.

Dehn's algorithm solves the word problem in word hyperbolic groups, reducing any representative of the identity to the empty word through a sequence of strictly length reducing substitutions.

Automatic and biautomatic groups

G is **automatic** if $\exists$ a regular set L of representative words over X, and $k \in \mathbb{N}$ s.t. any $v, w \in L$ with $|v^{-1}w|_G \leq 1$ k-fellow travel.
Then (L, k) is an **automatic structure** for G.

L is regular if '$w \in L$?' can be decided by reading w sequentially using bounded memory, i.e. with a finite state automaton, FSA.



Two words v, w **k-fellow travel** ($v \sim_k w$) if the paths they trace out from the origin of $\Gamma(G, X)$ stay at most k-apart through their length.

Definition of automaticity is independent of generating set, class is closed under finite extensions, direct products, free products, and more.

G is **shortlex automatic** if L can be chosen to be the minimal reps of group elements under the shortlex word order.

Biautomatic groups

G is **biautomatic** if G has an automatic structure (L, k) such that whenever $(v, w) \in L$ with $wv^{-1} =_G x \in X$, the paths traced out by w and xv k-'fellow-travel' as well.

Automatic groups have soluble word problem (soluble in quadratic time).

Biautomatic groups have soluble conjugacy problem (soluble in at worst doubly exponential time).

Artin groups of large type, what was known?

Artin groups of large and extra-large type were introduced and studied in:

K.I. Appel and P.E. Schupp, Artin groups and infinite Coxeter groups, Invent. Math. 72 (1983) 201–220,
K.I.Appel, On Artin groups and Coxeter groups of large type, Contemporary Mathematics 33 (1984) 50–78.

Using small cancellation techniques, they were proved to have soluble word and conjugacy problems. Parabolic subgroups $\langle x_i, i \in J \subseteq [1,n] \rangle$ were proved to embed as Artin subgroups, subgroups $\langle x_i^2 \rangle$ to be free.

(Bi)automaticity of Artin groups?

Artin groups of spherical type were proved biautomatic by Charney (1992), by direct construction of an appropriate structure.

Artin groups of extra large type were proved biautomatic by Peifer (1994).

3-generator Artin groups of large type, and many others of large type were proved biautomatic by McCammond and Wise (2000). Non-standard presentations of these groups satisfy small cancellation conditions that allow application of a result by Gersten and Short (1990).

Right angled Artin groups were proved biautomatic by Hermiller and Meier (1995), by direct construction.

It is still unknown whether all Artin groups are biautomatic. Shortlex (bi)automaticity was previously open for all except right angled groups.

Our results

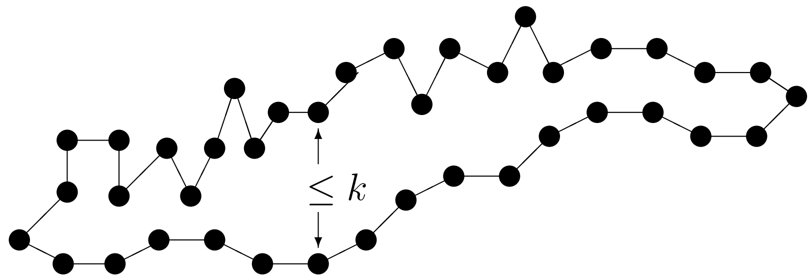
Theorem (Holt, Rees, PLMS 2011) Suppose that G is an Artin group of large type, defined over its natural generator set X. Then G is shortlex automatic.

As a consequence of this result (and its proof), we have

- easy descriptions of the geodesics and shortlex minimal geodesics in G,
- a straightforward algorithm to rewrite words into that form, and in particular to solve the word problem.
- We can deduce that parabolic subgroups are (quasi-)convex in G.

And so we can do more . . .

Theorem (Holt, Rees, PLMS 2011) An Artin group of large type in its standard presentation satisfies the falsification by fellow traveller condition. Hence the set of all geodesics is regular.



The **falsification by fellow traveller** condition (FFTP) is satisfied if, for some k, any non-geodesic word asynchronously k-fellow travels with a shorter representative of the same element. This condition is proved by Neumann and Shapiro (1995) to imply regularity of the set of all geodesics.

A further application

Theorem (Holt, Rees, preprint 2011)

Artin groups of extra large type satisfy the rapid decay condition.

A finitely generated group G satisfies **rapid decay** if the operator norm $||.||_*$ for the group algebra $\mathbb{C}G$ is bounded by a constant multiple of the Sobolev norm $||.||_{s,r,\ell}$, for some length function ℓ on G.

Rapid decay is relevant to the Novikov and Baum-Connes conjectures.

Explaining the notation for rapid decay

We define, for $\phi \in \mathbb{C}G$,

$$||\phi||_* = \sup_{\psi \in \mathbb{C}G} \frac{||\phi * \psi||_2}{||\psi||_2}, \quad \phi * \psi(g) = \sum_{h \in G} \phi(h)\psi(h^{-1}g),$$

$$||\psi||_2 = \sqrt{\sum_{g \in G} |\psi(g)|^2}, \quad ||\phi||_{2,r,\ell} = \sqrt{\sum_{g \in G} |\phi(g)|^2 (1 + \ell(g))^{2r}}.$$

A function $\ell : G \to \mathbb{R}$ is a length function if

$$\ell(1_G) = 0, \quad \ell(g^{-1}) = \ell(g), \quad \ell(gh) \leq \ell(g) + \ell(h), \quad \forall g, h \in G.$$

Proving it all. Our approach

We build on the work of Mairesse and Matheus, who characterised geodesics in dihedral Artin groups.

We shall see that in an Artin group of large type a word that is not shortlex minimal can be reduced using a sequence of reductions within 2-generator subwords.

In some sense this is analogous to the reduction process that solves the word problem in Coxeter groups (Tits' algorithm).

Coxeter groups

For a Coxeter group

$$G = \langle a_1, \ldots, a_n \mid a_i^2 = 1, {}_{\mathsf{m_{ij}}}(a_i, a_j) = {}_{\mathsf{m_{ij}}}(a_j, a_i) \quad \text{for each} \quad i \neq j \rangle,$$

Tits' algorithm solves the word problem in quadratic time, and uses the braid relations ${}_{\mathsf{m_{ij}}}(a_i, a_j) = {}_{\mathsf{m_{ij}}}(a_j, a_i)$

Tits' algorithm:

First replace any a_i^{-1} in w by a_i.

Then, while some application of a sequence of braid relations to w produces a subword a_i^2, delete that subword and repeat.
If $w \to \epsilon$, then return '$w =_G 1$'; otherwise return '$w \neq_G 1$'.

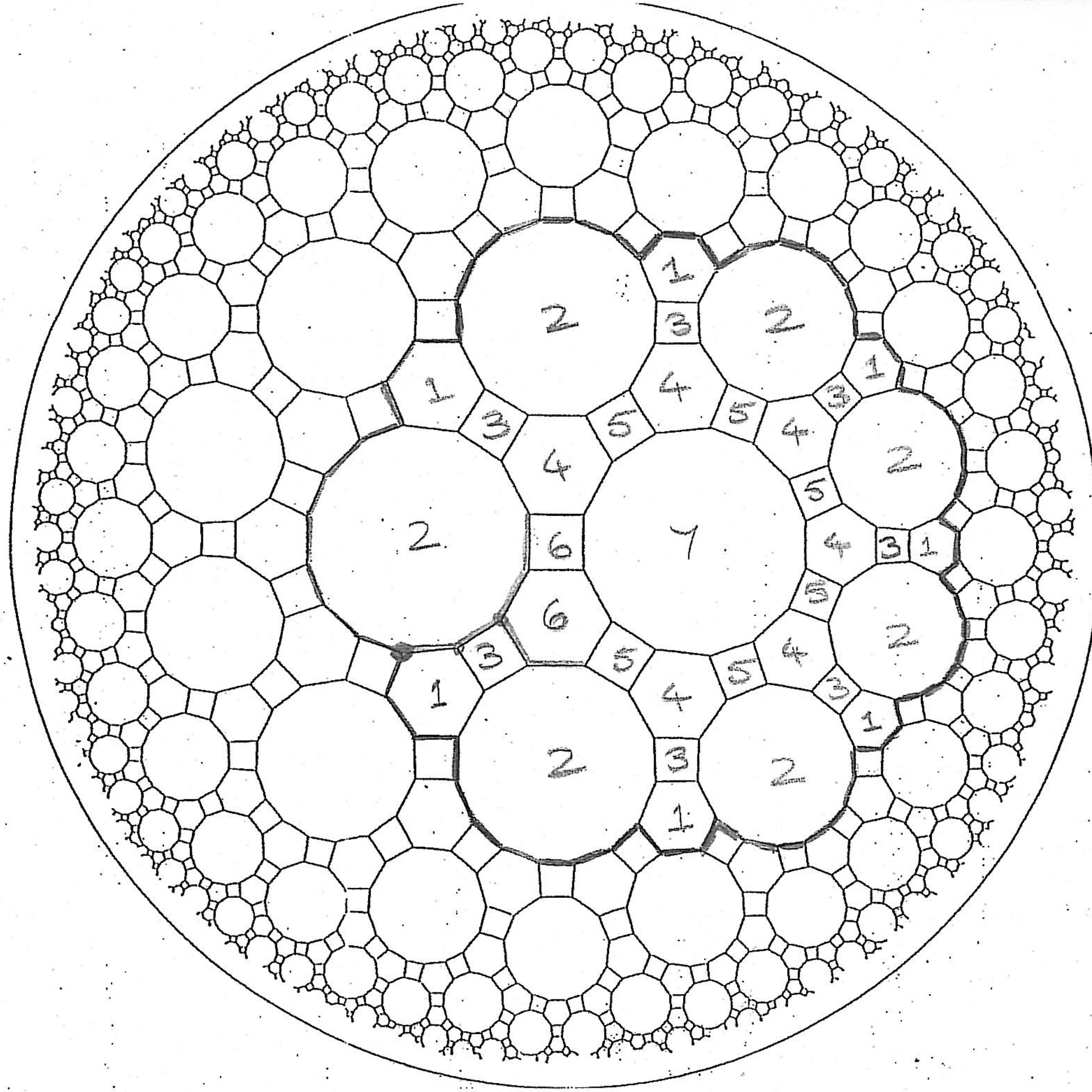
At every step in the reduction process w_{i+1} is no longer than w_i, and contains no generator not in w_i.

Example:

$$
\begin{aligned}
\langle a,b,c \;\; | \;\; & a^2=b^2=c^2=(ab)^3=(bc)^7=(ac)^2=1\rangle \\
(abacbcbcbc)^7 \to\; & (babcbcbcbc)^7 \\
\to\; & (bacbcbcbcc)^7 \to (bacbcbcb)^7 \to b(acbcbc)^7 b \\
\to\; & b(cabcbc)^7 b \to bc(abcb)^7 cb = bcab(cbab)^6 cbcb \\
\to\; & bcab(caba)^6 cbcb \\
\to\; & bcab(acba)^6 cbcb \to bcaba(cb)^6 acbcb \\
\to\; & bcbab(cb)^7 cabcb = bcba(bc)^7 abcb \to \epsilon
\end{aligned}
$$

We can see this reduction graphically (next slide).

Notice that although this group is in fact word hyperbolic the presentation we're using isn't a Dehn presentation and so Dehn's algorithm isn't valid. No relator intersects the input word in more than half of its length. We cannot reduce w with substitutions that reduce length at every stage.


1
2
3
2
1
4
1
3
5
5
3
4
4
2
5
2
6
7
4
3
1
5
6
2
3
5
5
4
1
4
3
1
2
3
2
1

Back to Artin groups . . .

Recall we have

$$G = \langle a_1, \ldots, a_n \mid \overbrace{a_i a_j a_i \cdots}^{\mathsf{m_{ij}}} = \overbrace{a_j a_i \cdots}^{\mathsf{m_{ij}}}, \quad i < j, \mathsf{m_{ij}} \in \mathbb{N} \cup \infty \rangle.$$

We write ${}_m(a, b)$ for the alternating product $aba \cdots$ of length m, and $(a, b)_m$ for $\cdots bab$.

Large type means that $\mathsf{m_{ij}} \geq 3$. We may assume that some $\mathsf{m_{ij}}$ are finite, since otherwise the group is free.

Dihedral Artin groups

The geodesics of all dihedral Artin groups have been studied by Mairesse and Mathéus. Let DA_m be the dihedral Artin group

$$DA_m = \langle a, b \mid {}_m(a,b) = {}_m(b,a) \rangle.$$

For any word w over a, b, we define $p(w)$, $n(w)$ as follows:

$p(w)$ is the minimum of m and the length of the longest positive alternating subword in w,

$n(w)$ is the minimum of m and the length of the longest negative alternating subword in w.

Example:

For $w = aba^{-1}baba^{-1}bababa^{-1}b^{-1}$, in DA_3, we have $p(w) = 3$, $n(w) = 2$. The following theorem tells us that w is non-geodesic.

Theorem (Mairesse,Mathéus, 2006)

In a dihedral Artin group DA_m, a word w is geodesic iff $p(w)+n(w) \leq m$, and is the unique geodesic representative of the element it represents if $p(w) + n(w) < m$.

The **Garside element** Δ in a dihedral Artin group is used in rewriting. Δ is represented by ${}_m(a, b)$ and either Δ or Δ^2 is central (depending on whether m is even or odd. If m is odd, Δ conjugates a to b.

We write δ for the permutation of $\{a, b, a^{-1}, b^{-1}\}^*$ induced by conjugation by Δ.

Examples:

In DA_3, $\Delta = aba =_G bab$. $a^\Delta = b$, and $\delta(ab^3a^{-1}) = ba^3b^{-1}$

In DA_4, $\Delta = abab =_G baba$. $a^\Delta = a$, and $\delta(ab^3a^{-1}) = ab^3a^{-1}$

Critical words in dihedral Artin groups.

We call a geodesic word v critical if it has the form

$$ {}_p(x,y)\xi(z^{-1},t^{-1})_n \quad \text{or} \quad {}_n(y^{-1},x^{-1})\delta(\xi)(t,z)_p, $$

where $\{x,y\} = \{z,t\} = \{a,b\}$, $p = p(v)$, $n = n(v)$. (We add an extra condition when p or n is zero.)

We define an involution τ on the set of critical words that swaps the words in each pair (as above). That $w =_G \tau(w)$ follows from the equations

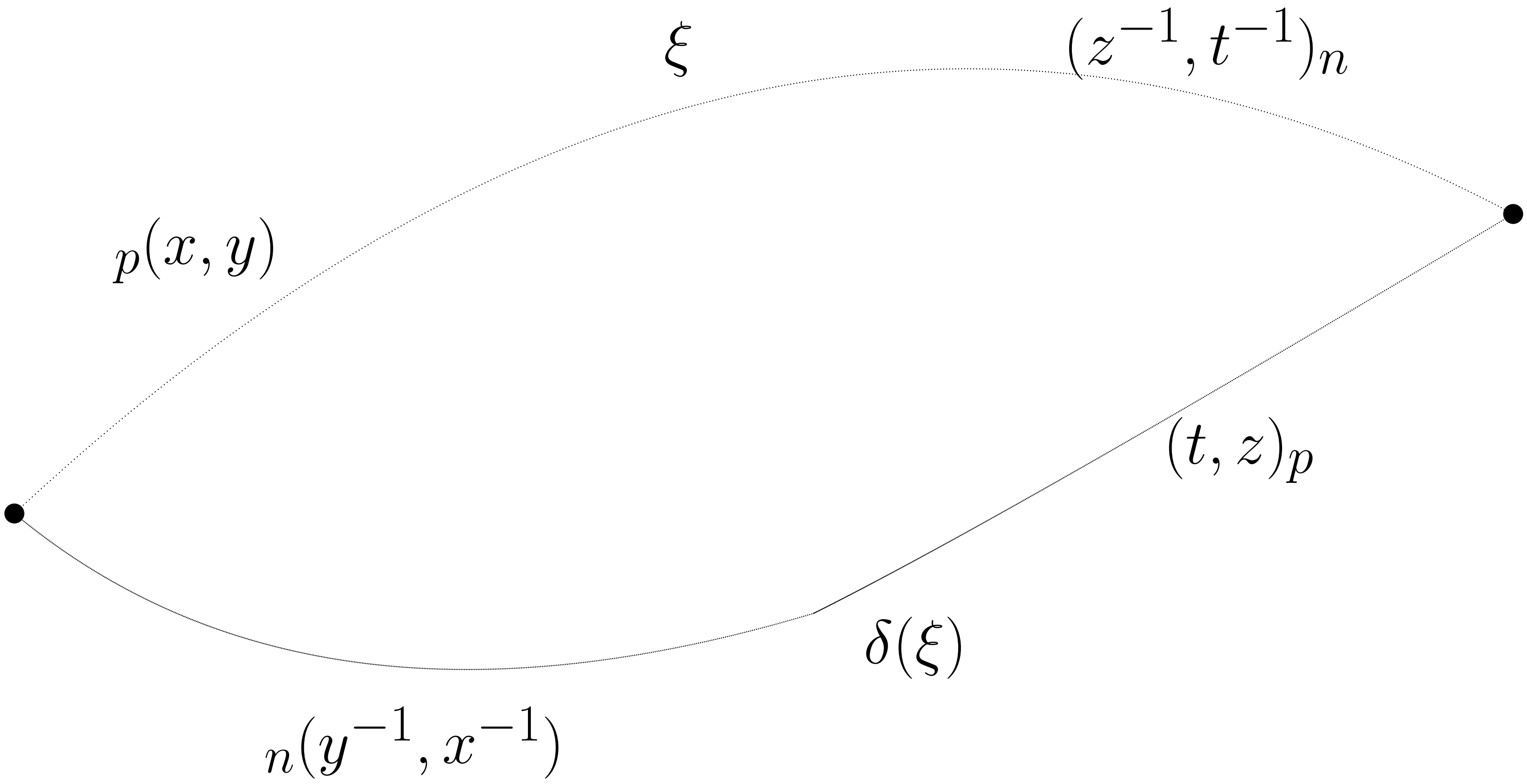
$$ {}_p(x,y) =_G {}_n(y^{-1},x^{-1})\Delta, \qquad \Delta\xi =_G \delta(\xi)\Delta, \quad \Delta(z^{-1},t^{-1})_n =_G (t,z)_p $$

The two words begin and end with different generators.

We call application of τ to a critical subword of w a **τ-move** on w.

We'll use τ-moves in much the same way that Tits' algorithm for Coxeter groups uses the braid relations.

Two critical words related by a τ-move.



Example:

In $G = DA_3$, aba^{-1} is critical, and $\tau(aba^{-1}) = b^{-1}ab$.

Applying that τ-move 2 (but not 3) times to the non-geodesic word w we met before,, we see that

$$w = (aba^{-1})b(aba^{-1})bab(aba^{-1})b^{-1} \to (aba^{-1})b(b^{-1}ab)bab(b^{-1}ab)b^{-1},$$

and the final word freely reduces to $abbbaa$, which is geodesic.

It is straightforward to derive the following from Mathéus and Mairesse' criterion for geodesics.

Theorem (Holt, Rees, PLMS 2011)

If w is freely reduced over $\{a, b\}$ then w is shortlex minimal in DA_m unless it can be written as $w_1w_2w_3$ where w_2 is critical, and $w' = w_1\tau(w_2)w_3$ is either less than w lexicographically or not freely reduced.

Applying τ moves in Artin groups of large type.

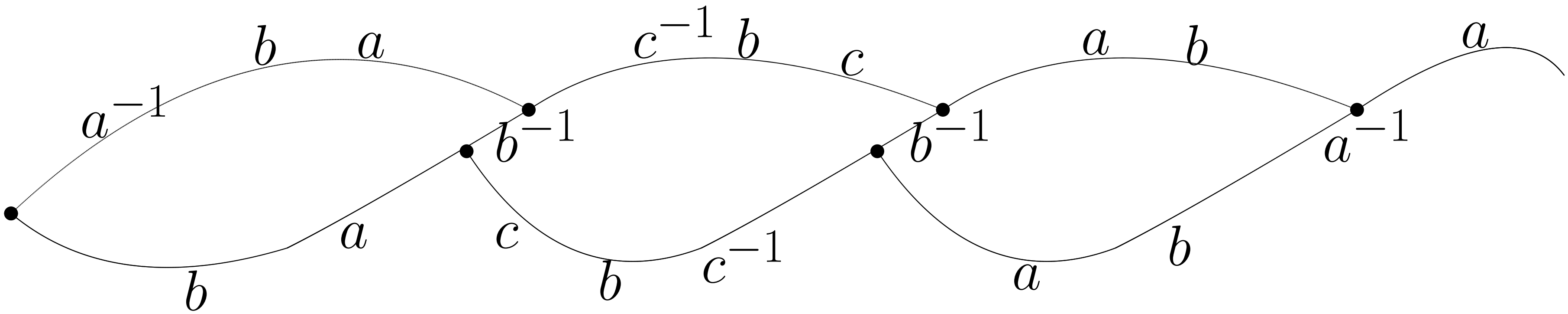
When we have more than 2 generators, we reduce to shortlex minimal form using sequences of τ-moves, each on a 2-generator subword.

Example:

$$G = \langle a, b, c \mid aba = bab,\ aca = cac,\ bcbc = cbcb \rangle$$

First consider $w = a^{-1}bac^{-1}bcaba$. The 2 generator subwords $a^{-1}ba$, $c^{-1}bc$, aba are all geodesic in the dihedral Artin subgroups (in fact also in G). The two maximal a, b subwords are critical in DA_3. Applying a τ-move to the leftmost critical subword creates a new critical subword, to which we can then apply a τ-move.

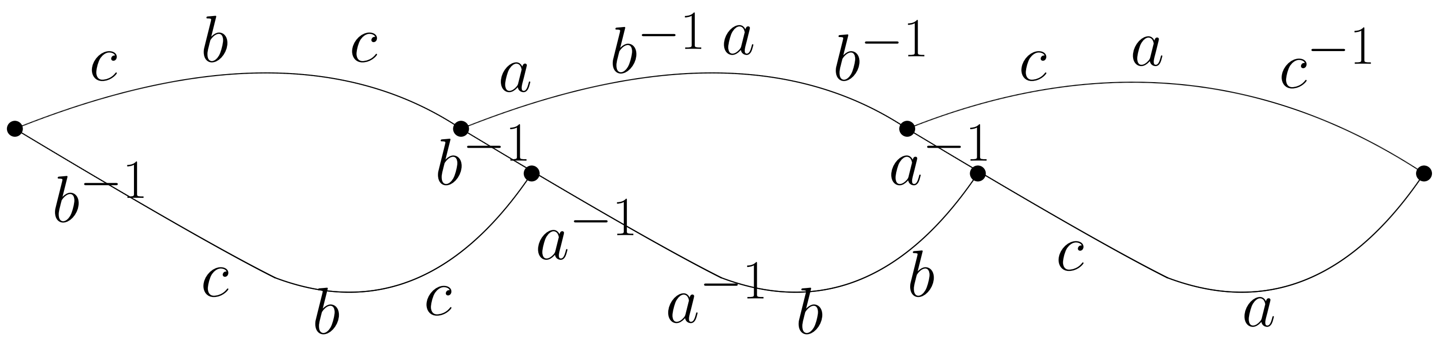
In fact, a sequence of 3 τ-moves transforms w to a word that is not freely reduced. The free reduction is then $bacbc^{-1}ab$,



Reducing $a^{-1}bac^{-1}bcaba$.

We call a sequence of τ-moves like this a **rightward length reducing sequence.**

Now consider $w = cbcab^{-1}ab^{-1}cac^{-1}$, in which cac^{-1} is critical. Applying a τ-move to this critical subword creates a new critical subword, $ab^{-1}ab^{-1}a^{-1}$, to which we can then apply a further τ-move. After one more τ-move, w is transformed to the word $w' = b^{-1}cbca^{-1}a^{-1}bbca$, of the same length as w but preceding w lexicographically.



We call a sequence like this a **leftward lex reducing sequence.**

A shortlex automatic structure

Let L be the regular set of words that excludes w iff it admits **either** a rightward length reducing sequence of τ-moves **or** a leftward lex areducing sequence of τ-moves. Certainly L contains all shortlex minimal reps. To prove our theorem we need the following.

Proposition (Holt, Rees)
If $w \in L$ but $wg \notin L$ then
either (1) a single rightward sequence of τ-moves on w transforms w to a word $w'g^{-1}$, and $w' \in L$
or (2) a single leftward sequence of τ-moves on wg transforms wg to a word w'' less than wg, and $w'' \in L$.

NB: w and w', w'' k=fellow travel, for $k = 2\max\{\mathsf{m_{ij}} : \mathsf{m_{ij}} < \infty\}$.

Crucial to the proof of the proposition:

Application of a single sequence of τ-moves to a word preserves the sequence of pairs of generators that appear in successive, overlapping, maximal 2-generator subwords.

We can check that this is valid for the reductions

$$a^{-1}bac^{-1}bcaba \to bacbc^{-1}ab \quad \text{and}$$
$$cbcab^{-1}ab^{-1}cac^{-1} \to b^{-1}cbca^{-1}a^{-1}bbca$$

in

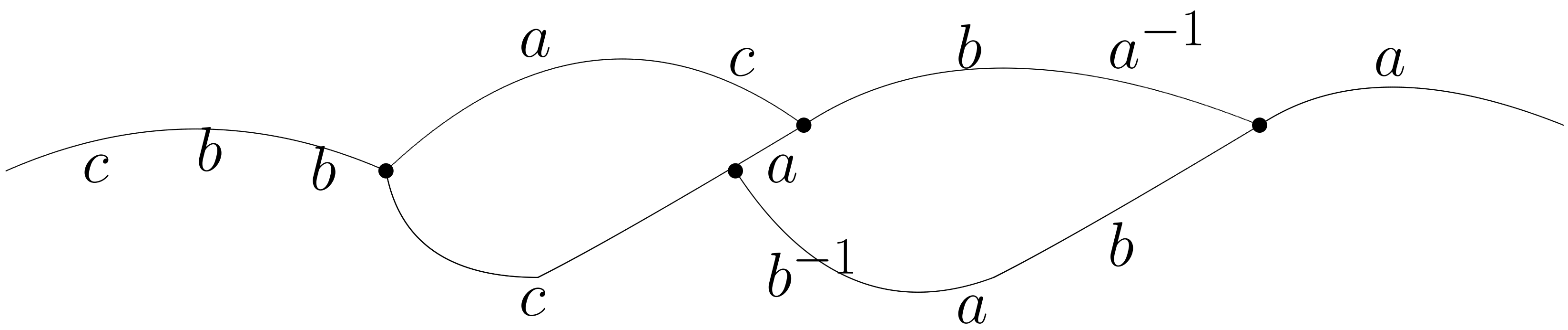
$$G = \langle a, b, c \mid aba = bab,\ aca = cac,\ bcbc = cbcb \rangle.$$

But this is a consequence of large type.

The proposition fails without large type

$$\text{Let} \quad G = \langle a, b, c \mid aba = bab,\ ac = ca,\ bcb = cbc \rangle.$$

Then $w = cbbacba^{-1} \in W$, but wb^{-1} admits a rightward length reducing sequence to $w' = cbbcb^{-1}a$, which then reduces lexicographically, to $w'' = b^{-1}cbbca$. And $w''b <_{\text{slex}} w$.



In the rightward reduction of wb^{-1} a sequence of 5 overlapping max 2-gen subwords collapses to a sequence of 2.

Applying the proposition to prove G shortlex automatic

We define a map $\rho : X^{\pm *} \to L$ as follows, for words w, generators g:-

$\rho(w) := w$ if $w \in L$.

Let $w \in L, wg \notin L$.

If wg is freely reduced, then
$\rho(wg) := w'$ in case (1) of the proposition, $\rho(wg) := w''$ in case (2),
and otherwise $\rho(wg)$ is the free reduction of wg.

More generally $\rho(wg) := \rho(\rho(w)g)$.

We prove that $\rho(wgg^{-1}) = w, \quad \rho(w(x_i, x_j)_{\mathsf{m_{ij}}}) = \rho(w(x_j, x_i)_{\mathsf{m_{ij}}})$, and so whenever $v =_G w$, $\rho(v) = \rho(w)$. So L contains a unique representative for each element of G, consists **only** of shortlex minimal reps.

Verifying FFTP

It is enough to verify FFTP for minimally non-geodesic words vg. This is trivial to do if vg freely reduces.
So we may suppose that vg is freely reduced, i.e. that $\mathrm{l}[v] \neq g^{-1}$. For such a word vg,
v' is a geodesic rep for vg iff $v'g^{-1}$ is a geodesic rep for v.

So the FFTP follows from the following.

Proposition (Holt, Rees)

$\exists k$ such that if v, w are geodesics in G with $\mathrm{l}[v] \neq \mathrm{l}[w]$, and $v =_G w$, then

$$\exists w', \quad w' =_G v, \quad \mathrm{l}[w'] = \mathrm{l}[w], \quad w' \sim_k v.$$

Proof: We examine the process of reduction to shortlex minimal form.

Verifying rapid decay

We verify rapid decay by verifying the property

$$\forall \phi, \psi \in \mathbb{C}G, k, l, m \in \mathbb{N},$$
$$|k-l| \leq m \leq k+l, \Rightarrow ||(\phi_k * \psi_l)_m||_2 \leq P(k)||\phi_k||_2||\psi_l||_2.$$

where ϕ_k means the restriction of ϕ to elements of word length k, and we similarly interpret the subscripted l and m.

Verification of the property requires understanding of the possible ways in which elements of length m can be factorised as products of elements of length k, l.