Subgroup Distortion in Wreath Products of Cyclic Groups

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Outline

Subgroup Distortion

Definition (Gromov)

For a finitely generated group $G = \langle T \rangle$ and a f.g. subgroup $H = \langle S \rangle$, the distortion function of H in G is

$$\Delta_H^G(l) = \max\{|w|_S : w \in H, |w|_T \leq l\}.$$

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- For $f, g : \mathbb{N} \to \mathbb{N}$, we say that $f \preceq g$ if there exists an integer $C > 0$ such that

$$f(l) \leq Cg(Cl) + Cl$$

for all $l \geq 0$.

Examples of Subgroup Distortion

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- $$a^{2^l} = b^l a b^{-l}$$

Wreath Products

Let A and B be any groups.

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- The (left) action $\circ$ of B on W by automorphisms is given by: $(g \circ w)(x) = w(xg)$, for any $w \in W, g \in B$ and $x \in B$.
- For $g_1, g_2 \in B, w_1, w_2 \in W$ we have that $(w_1 g_1)(w_2 g_2) = (w_1(g_1 \circ w_2))(g_1 g_2)$.

Example: $\mathbb{Z} \text{ wr } \mathbb{Z}$

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- The group W is a free module with one generator a over the group ring $\mathbb{Z}[\langle b \rangle]$.

Word Metric [Cleary, Taback]

- Arbitrary element of $A \text{ wr } \mathbb{Z}$ may be written in a normal form as

$$(b^{\iota_1} \circ u_1) \cdots (b^{\iota_N} \circ u_N)(b^{-\epsilon_1} \circ v_1) \cdots (b^{-\epsilon_M} \circ v_M) b^t$$

where $0 \leq \iota_1 < \cdots < \iota_N, 0 < \epsilon_1 < \cdots < \epsilon_M$, and $u_1, \ldots, u_N, v_1, \ldots, v_M$ are elements in $A - \{1\}$.

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- The length is given by the formula

$$\sum_{i=1}^{N} |u_i|_A + \sum_{i=1}^{M} |v_i|_A + \min\{2\epsilon_M + \iota_N + |t - \iota_N|, 2\iota_N + \epsilon_M + |t + \epsilon_M|\}.$$

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- Its length equals the sum of lengths of a^{-3}, a^4 and a^2 in $\langle a\rangle$ plus $\min\{2\epsilon_2 + \iota_1 + |t - \iota_1|, 2\iota_1 + \epsilon_2 + |t + \epsilon_2|\} = 2\epsilon_2 + \iota_1 + |t - \iota_1|$.

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- This equals $3 + 4 + 2 + 11$ and is recognized by

$$(b^{-1}a^4b)(b^{-2}a^2b^2)(b^5a^{-3}b^{-5})b^3 = b^{-1}a^4b^{-1}a^2b^7a^{-3}b^{-2}.$$

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- The minimum corresponds to a path in the Cayley graph of $\langle b\rangle$ which starts at 1, passes b^{-1}, b^{-2}, b^5 and ends at b^3.

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- Any $u = wg \in A \operatorname{wr} B$ can be expressed in canonical form as $(b_1 \circ a_1) \ldots (b_r \circ a_r)g$ where $g \in B, 1 \neq a_j \in A, b_j \in B$ and for $i \neq j$ we have $b_i \neq b_j$.

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- Define the norm of any such representative w of W by

$$||w||_A = \sum_{j=1}^{r} |a_j|_S.$$

Word Metric/Applications

Theorem

For any element $u = wg \in A \; wr \; B$*, we have that*

$$|wg|_{S,T} = ||w||_A + \text{reach}(u)$$

where $u = (b_1 \circ a_1) \ldots (b_r \circ a_r)g$ *is the canonical form above.*

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- As an application of the formula, we can show that the group $\mathbb{Z}_2 \text{ wr } \mathbb{Z}^2$ contains distorted subgroups.
- This is interesting in contrast to the case of $\mathbb{Z}_2 \text{ wr } \mathbb{Z}$ which has no effects of subgroup distortion.

Distortion in $\mathbb{Z}_2$ wr $\mathbb{Z}^2$

- Let $G = \mathbb{Z}_2 \text{ wr } \mathbb{Z}^2 = \text{gp}\langle a, b, c \rangle = W \lambda \mathbb{Z}^2$.

Distortion in $\mathbb{Z}_2 \text{ wr } \mathbb{Z}^2$

- Let $G = \mathbb{Z}_2 \text{ wr } \mathbb{Z}^2 = \text{gp}\langle a, b, c\rangle = W \lambda \mathbb{Z}^2$.
- $W = \bigoplus_{g \in \mathbb{Z}^2} \langle g \circ a\rangle$ is a free module over $\mathbb{Z}_2[\mathbb{Z}^2]$. Therefore, we may think of W as being the Laurent polynomial ring in two variables, say, x for b and y for c.

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- Let $H = \mathrm{gp}\langle b, c, w\rangle$ where $w = [a, b] = (1 - x)a$.

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- Let $H = \text{gp}\langle b, c, w\rangle$ where $w = [a, b] = (1 - x)a$.
- Then $H \cong G$.

Distortion in $\mathbb{Z}_2 \text{ wr } \mathbb{Z}^2$

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$$f_l(x) = \sum_{i=0}^{l-1} x^i \text{ and } g_l(x) = (1-x)f_l(x).$$

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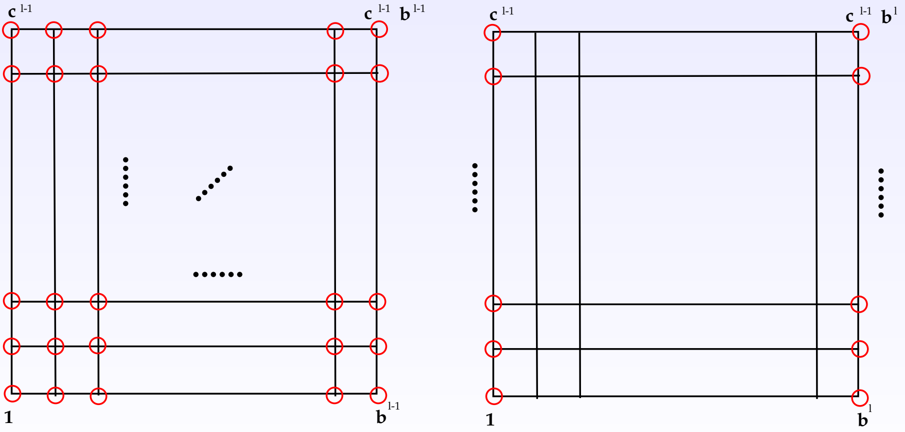
- Its length in H is at least $l^2 + l^2$ since the support of it has cardinality l^2, and the length of arbitrary loop going through l^2 different vertices is at least l^2.

Distortion in $\mathbb{Z}_2$ wr $\mathbb{Z}^2$

The l^2 vertices (left) and the rectangle with perimeter $2l + 2(l-1)$ (right)



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- Therefore, $|f_l(x)f_l(y)w|_G = 2l + 2(l-1) + 2l$.
- This is because the shortest path in Cay($\mathbb{Z}^2$) starting at 1, passing through $1, c, \ldots, c^{l-1}$ and $b^l, cb^l, \ldots, c^{l-1}b^l$ and ending at 1 is given by traversing the perimeter of the rectangle, and so gives the length of $2(l-1) + 2l$.

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- This is because the shortest path in $\text{Cay}(\mathbb{Z}^2)$ starting at 1, passing through $1, c, \ldots, c^{l-1}$ and $b^l, cb^l, \ldots, c^{l-1}b^l$ and ending at 1 is given by traversing the perimeter of the rectangle, and so gives the length of $2(l-1) + 2l$.
- Therefore the subgroup H is at least quadratically distorted.

Main Theorem

Theorem

Let A be a finitely generated abelian group.

1. *For any finitely generated subgroup $H \leq A \text{ wr } \mathbb{Z}$ there exists $m \in \mathbb{N}$ such that the distortion of H in $A \text{ wr } \mathbb{Z}$ is*

$$\Delta_H^{A \text{ wr } \mathbb{Z}}(l) \approx l^m.$$

2. *If A is finite, then $m = 1$; that is, all subgroups are undistorted.*
3. *If A is infinite, then for every $m \in \mathbb{N}$, there is a 2-generated subnormal subgroup H of $A \text{ wr } \mathbb{Z}$ having distortion function*

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- Distortion in free metabelian groups is similar to that in wreath products because if $k \geq 2$ then $\mathbb{Z} \text{ wr } \mathbb{Z} \leq S_{k,2} \leq \mathbb{Z}^k \text{ wr } \mathbb{Z}^k$.

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- Every finitely generated abelian subgroup of $\mathbb{Z}^k \text{ wr } \mathbb{Z}$ is undistorted. [Guba, Sapir]

More Motivation

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- For example, the map defined on generators by $b \mapsto b, a \mapsto [a, b]$ extends to an embedding, and the image is a quadratically distorted subgroup.
- Thus there is a distorted embedding of $\mathbb{Z} \text{ wr } \mathbb{Z}$ into Thompson's group F.
- The group $\mathbb{Z} \text{ wr } \mathbb{Z}$ is the smallest metabelian group which embedds to itself as a normal distorted subgroup: For any metabelian group G, if there is an embedding $\phi : G \to G$ such that $\phi(G) \trianglelefteq G$ and $\phi(G)$ is a distorted subgroup in G, then there exists some subgroup H of G for which $H \cong \mathbb{Z} \text{ wr } \mathbb{Z}$.

Subgroups "with b"

From now on A is finitely generated abelian and $\mathbb{Z} = \langle b \rangle$.

- Any finitely generated subgroup in $A \text{ wr } \mathbb{Z} = W \lambda \langle b \rangle$ can be generated by elements $w_1 b^t, w_2, \ldots, w_s$ where $w_i \in W$.

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- It turns out that having b as a generator is more convenient for computations than $w_1 b^t$.
- We call $H \leq A \operatorname{wr} \mathbb{Z}$ "a subgroup with b" if the generators of H may be given by $b, w_1, \ldots, w_s$ for $w_i \in W$.

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- It turns out that having b as a generator is more convenient for computations than $w_1 b^t$.
- We call $H \leq A \text{ wr } \mathbb{Z}$ "a subgroup with b" if the generators of H may be given by $b, w_1, \ldots, w_s$ for $w_i \in W$.
- If H is a f.g. subgroup of $A \text{ wr } \mathbb{Z}$ not contained in W, then the distortion of H in $A \text{ wr } \mathbb{Z}$ is equivalent to the distortion of a subgroup H' in $A^t \text{ wr } \mathbb{Z}$ with b.

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Reducing to the Case where A is free abelian

- Consider $G = A \text{ wr } \mathbb{Z} = A \text{ wr } \langle b \rangle$.
- Let H be a finitely generated subgroup of G.
- Then there exists r so that the distortion of H in G is equivalent to that of a finitely generated subgroup in $\mathbb{Z}^r \text{ wr } \mathbb{Z}$.
- Therefore, it suffices to study subgroups H of $\mathbb{Z}^r \text{ wr } \mathbb{Z}$ with b.

"Special" Subgroups

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- We call a subgroup H of G "special" if H can be generated by elements $b, w_1, \ldots, w_k$ where each w_i is in the normal closure of only one a_i.

"Special" Subgroups

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- We call a subgroup H of G "special" if H can be generated by elements $b, w_1, \dots, w_k$ where each w_i is in the normal closure of only one a_i.
- Let $H \leq \mathbb{Z}^r \text{ wr } \mathbb{Z}$ be a special subgroup with generators $b, w_1, \dots, w_k$.Then H is isomorphic to $\mathbb{Z}^k \text{ wr } \mathbb{Z}$.

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- Let H be a subgroup of $\mathbb{Z}^r \text{ wr } \mathbb{Z}$ with b. Then the distortion of H in $\mathbb{Z}^r \text{ wr } \mathbb{Z}$ is equivalent to the distortion of a special subgroup.

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- Further, we may assume without loss of generality that each H_i is a tame subgroup.

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- We will define a function $S : R[x] \to \mathbb{R}^+$ which takes any $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ to $S(f) = \sum_{i=0}^{n} |a_i|$.

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- For any $h \in R[x]$, we define the distortion of the polynomial h from $\mathbb{N}$ to $\mathbb{N}$ by

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- The distortion does not depend on the constant c, up to equivalence, and so we will consider $\Delta_h(l)$.

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- Then
$$\Delta_h(l) \approx \Delta_H^{\mathbb{Z} \text{ wr } \mathbb{Z}}(l).$$
- So we would like to be able to explicitly compute the distortion of any polynomial.

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- The distortion of h with respect to the ring of polynomials over $\mathbb{Z}, \mathbb{R}$, or $\mathbb{C}$ is bounded from below by $l^{\kappa+1}$, up to equivalence, where h has a complex root of multiplicity κ and modulus one.
- Obtaining upper bounds requires linear algebra, but it can be shown that $\Delta_h(l) \approx l^{\kappa+1}$ where κ is the maximal multiplicity of any complex root of h with modulus one.

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- This proves that the distortion of H is equivalent to a polynomial.

Describing 2-generated distorted subgroups in $\mathbb{Z} \text{ wr } \mathbb{Z}$

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- The subgroup

$$H = \langle b, [\cdots [a, b], b], \cdots, b] \rangle,$$

where the commutator is $(m - 1)$-fold.

Thank you!