One-ended subgroups of graphs of free groups

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Questions

Gromov wrote

> *'[O]ne may suspect that there exist word hyperbolic groups* Γ *with arbitrarily large* $\dim \partial\Gamma$ *(here, large is* ≥ 1*) where every proper subgroup is free.'*

Question (The Surface Subgroup Problem)

Does every one-ended hyperbolic group have a surface subgroup?

Question (The Non-free Subgroup Problem)

Does every one-ended non-surface hyperbolic group have a non-free subgroup of infinite index?

Context

- The Surface Subgroup Problem is motivated by the famous case that Γ is the fundamental group of a closed hyperbolic 3-manifold, recently answered affirmatively by Kahn–Markovic.
- For 3-manifolds, The Non-Free Subgroup Problem is just as hard, by Scott's Theorem that 3-manifold groups are coherent and Dehn's Lemma.
- These questions remain open for some very easy groups, eg the 'Baumslag double' of a free group

$$D(w) = F *_{\langle w \rangle} F \ .$$

  $D(w)$ is hyperbolic when w is not a proper power and one-ended when F does not split as $A * B$ with w contained in a free factor. These last sort of examples will concern us today.

Known results on Surface Subgroups

- Calegari proved that a hyperbolic graph of free groups with cyclic edge groups contains a surface subgroup if $b_2 > 0$. This reduces the problem to finding subgroups with second homology.
- Gordon–W. found necessary conditions for many doubles to contain surface subgroups.
- Kim–W. found further necessary conditions.
- Kim–Oum found surface subgroups in all doubles of free groups *of rank two*.

Non-free subgroups

The main result of this talk is to solve the Non-Free Subgroup Problem for graphs of free groups with cyclic edge groups.

Theorem (W.)

If a one-ended hyperbolic group $\Gamma \cong F *_{\langle w \rangle} F'$ or $\Gamma \cong F*_{\langle w \rangle}$, where F, F' are free groups, then either Γ is the fundamental group of a closed surface or Γ has a finitely generated non-free subgroup of infinite index.

Since these are the only non-trivial cases, this shows that one cannot construct such an example by gluing along cyclic subgroups.

Corollary

If Γ is hyperbolic and splits over a virtually cyclic subgroup then either Γ is the fundamental group of a surface or Γ has a finitely generated non-free subgroup of infinite index.

Other classes of groups

Limit groups arise in the study of algebraic geometry over free groups. They either contain $\mathbb{Z}^2$ or they are hyperbolic, and they always split over a cyclic subgroup. Therefore:

Corollary

Any limit group is either the fundamental group of a surface or contains a finitely generated non-free subgroup of infinite index.

The class of *special groups* was introduced by Haglund and Wise. Limit groups and graphs of free groups are known to be (virtually) special. It should be possible to extend the main theorem to apply to special groups.

Peripheral structures

This result also has some bearing on the Surface Subgroup Problem. To state this, we need some definitions.

- A *peripheral structure* $[\underline{w}]$ on a free group F is a finite set of conjugacy classes of maximal cyclic subgroups $\langle w_i \rangle$.
- If $\widehat{F} \subseteq F$ is a subgroup, there is a natural pullback peripheral structure $[\underline{\hat{w}}]$ on $\widehat{F}$.
- A peripheral structure $[\underline{u}]$ on $\widehat{F}$ is *compatible* with $[\underline{w}]$ if $[\underline{u}] \subseteq [\underline{\hat{w}}]$.
- A pair $(F, [\underline{w}])$ is called *freely indecomposable*, or *one-ended*, if F does not split freely as $A * B$, in which the elements of $[\underline{w}]$ are conjugate into A or B.
- A pair $(F, [\underline{w}])$ is called *acylindrical* if F does not split as $A *_{\mathbb{Z}} B$ in which the elements of $[\underline{w}]$ are conjugate into A or B.

The poset of commensurability classes

- Given $(F, [\underline{w}])$, let $(H, [\underline{u}])$ and $(K, [\underline{v}])$ be non-abelian subgroups of F equipped with compatible peripheral structures.
- Let $[\hat{\underline{u}}]$ be the peripheral structure on $H \cap K$ induced by $[\underline{u}]$.
- Write
$$(H, [\underline{u}]) \leq (K, [\underline{v}])$$
if $|H : H \cap K| < \infty$ and $[\hat{\underline{u}}]$ is compatible with $[\underline{v}]$.
- Pairs $(H, [\underline{u}])$ and $(K, [\underline{v}])$ are *commensurable* if $(H, [\underline{u}]) \leq (K, [\underline{v}])$ and $(H, [\underline{u}]) \geq (K, [\underline{v}])$.
- Commensurability is an equivalence relation, and the relation $\leq$ descends to a partial order on equivalence classes.

Definition

Let $\mathcal{P}(\underline{w})$ be the poset of commensurability classes of non-abelian one-ended $(H, [\underline{u}])$ compatible with $(F, [\underline{w}])$.

Minimal elements

These techniques characterise surfaces among all free groups with peripheral structures.

Theorem (W.)

The commensurability class of a pair $(F, [\underline{w}])$ *is minimal if and only if* $(F, [\underline{w}]) \cong (\pi_1\Sigma, \partial\Sigma)$ *for* Σ *a compact surface.*

Proof.

Apply the main theorem to the double of F along $\underline{w}$. □

Could it be possible to find surface subgroups using Zorn's Lemma?

Question

Does every chain in $\mathcal{P}(\underline{w})$ *have a lower bound?*

Outline of the proof

- A Shenitzer-style theorem detects non-freeness.

Theorem (Folklore, cf. Shenitzer)

If a graph of groups with cyclic edge groups splits freely then some vertex group splits freely relative to its incident edge groups.

- The JSJ decomposition of Γ allows us to reduce to acylindrical pairs.

Theorem (Bowditch)

Any hyperbolic, one-ended Γ has a graph-of-groups decomposition in which every vertex group is acylindrical or a surface.

Outline of the proof, continued

- We prove a Local Theorem about acylindrical pairs.

Theorem (W.)

Suppose $(F, [\underline{w}])$ is acylindrical. If $\widehat{F} \subseteq F$ is a sufficiently deep finite-index subgroup of F and $\hat{w}_i \in \underline{\hat{w}}$ then $(\widehat{F}, [\underline{\hat{w}} \smallsetminus \hat{w}_i])$ is one-ended.

- Finally, we glue the pieces provided by the Local Theorem together carefully to produce a non-free subgroup of infinite index.

In the remainder of this talk, I will try to say something about the proof of the Local Theorem. The key tool is the Whitehead graph of $\underline{w}$.

Whitehead graphs

Let $\underline{w}$ be a set of non-trivial elements of F, written as cyclic words. The Whitehead graph is a combinatorial tool that allows us to recognise splittings of the pair $(F, [\underline{w}])$.

Definition

Fix an identification of the free group F with the fundamental group of a handlebody H. The set of words $\underline{w}$ can be represented by a 1-dimensional submanifold $N \subseteq H$. A basis B for F corresponds to a set of properly embedded discs that cut H into a ball. After cutting, each disc corresponds to two discs on the surface of the ball, and the submanifold consists of arcs between these discs. Crushing the discs to points, the resulting graph $W_B(\underline{w})$ is the *Whitehead graph* of $\underline{w}$.

An example

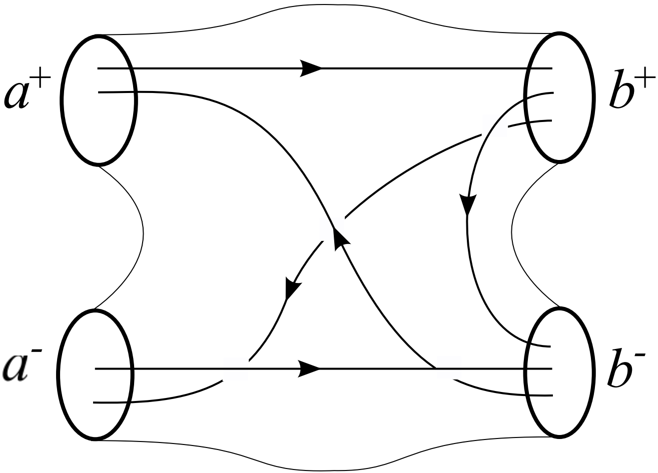

Figure: The word $b^{-1}aba^{-2}$ realised as a submanifold of a handlebody.

Recognising splittings

We can use Whitehead graphs to recognise one-ended and acylindrical pairs.

Lemma

If $(F, [\underline{w}])$ splits freely then $W_B(\underline{w})$ contains a cut vertex. If in addition B minimises the length of $\underline{w}$ then $W_B(\underline{w})$ is disconnected.

Lemma

If $W_B(\underline{w})$ has a separating pair of edges then $(F, [\underline{w}])$ splits cyclically. If $W_B(\underline{w})$ is separated by the removal of a vertex and an edge then either $W_B(\underline{w})$ has a separating pair of edges or B does not minimise the length of $\underline{w}$.

Splicing

We need to understand what happens to Whitehead graphs when we pass to a subgroup $\widehat{F}$ of finite index in F.

Definition (Manning)

If G_1, G_2 are graphs and $v_i \in G_i$ are vertices with equal valence then any graph G obtained by deleting the v_i and gluing the resulting edges according to some bijection is said to be obtained by *splicing*.

Lemma (Manning)

If $\widehat{F} \subseteq F$ is a subgroup of finite index and $\underline{\hat{w}}$ is the pullback of $\underline{w}$ to $\widehat{F}$ then for some suitable basis $\widehat{B}$ of $\widehat{F}$, the Whitehead graph $W_{\widehat{B}}(\underline{\hat{w}})$ is obtained by splicing finitely many copies of $W_B(\underline{w})$.

Splicing continued

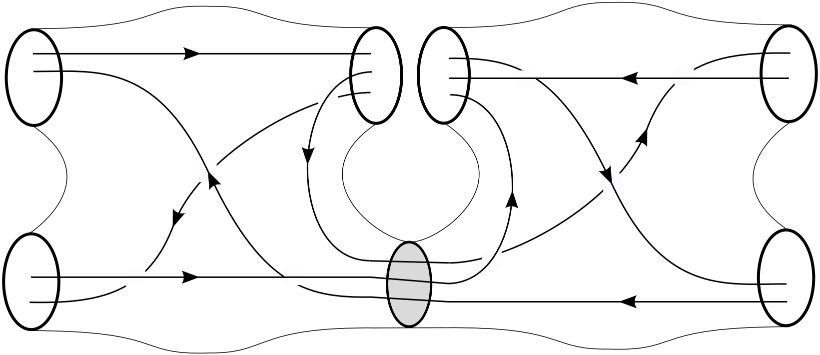

Figure: The lift of $b^{-1}aba^{-2}$ to a cover of degree two. Deleting the grey disc corresponds to splicing.

Splittings of finite-index subgroups

Manning's observation makes it easy to study finite-index subgroups combinatorially.

Remark

If G_1 and G_2 are connected graphs without cut vertices and G is obtained by splicing G_1 and G_2 then G is also connected without cut vertices.

Combining this with the Shenitzer-style theorem, we can reprove Stallings's Ends Theorem in the special case of graphs of free groups with cyclic edge groups.

Proposition

Let Γ be the fundamental group of a graph of free groups with cyclic edge groups and $\widehat{\Gamma}$ a subgroup of finite index. If $\widehat{\Gamma}$ splits freely then so does Γ.

Clean subgroups

We can finally explain the 'sufficiently deep' subgroups mentioned in the statement of the Local Theorem.

Definition

Fix a system of discs D in H representing a basis for F. A subgroup $\widehat{F}$ of finite index in F corresponds to a covering space $\widehat{H}$ of H. The space $\widehat{H}$ can be constructed by gluing together a finite set of copies $\{B_i\}$ of the ball $H \smallsetminus D$. The subgroup $\widehat{F}$ is *clean* if every component of the submanifold representing the pullback $\hat{w}$ intersects each ball B_i in a single arc.

An easy application of Marshall Hall's Theorem ensures that there are many clean subgroups.

Lemma

If $\overline{F}$ is any subgroup of finite index in F then there is a clean subgroup $\widehat{F}$ of finite index in $\overline{F}$.

Clean subgroups continued

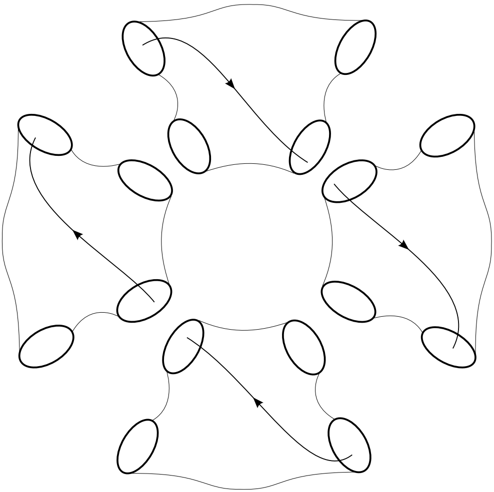

Figure: One component of the pullback of a word to a clean cover.

Proof of the Local Theorem

Theorem (Local Theorem)

Suppose $(F, [\underline{w}])$ is acylindrical. If $\widehat{F} \subseteq F$ is any clean subgroup of finite index in F and $\hat{w}_i \in \underline{\hat{w}}$ then $(\widehat{F}, [\underline{\hat{w}} \smallsetminus \hat{w}_i])$ is one-ended.

Proof.

Let B be chosen to minimise the length of $\underline{w}$ and let $W \equiv W_B(\underline{w})$. Then $W_{\hat{B}}(\underline{\hat{w}})$ is constructed by splicing together some copies $W_1, \ldots, W_n$ of W. Because $\widehat{F}$ is clean, $\hat{w}_j$ intersects each W_i in a single edge, so $W_{\hat{B}}(\underline{\hat{w}} \smallsetminus \hat{w}_i)$ is obtained by splicing together $W'_1, \ldots, W'_n$ where W'_j is W_j with a single edge deleted. Because $(F, [\underline{w}])$ is acylindrical, it follows that W'_j has no cut vertices, as required. □