Simplicial approximations of the Julia sets using Group Theory

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International Group Theory Webinar

Topological setting

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Finite automata (transducers).

Transducers

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An *automaton* over an alphabet X is a triple (Q, τ, π), where Q is a set (of *internal states*) and τ and π are maps

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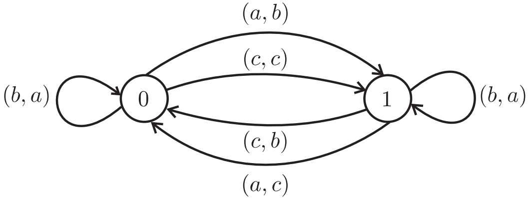
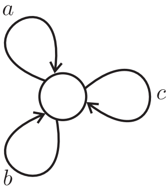
Dual Moore diagram





Iterating automata

Let $\mathcal{M}_0 = \mathcal{M}$, $f_0 = f$ and $\iota_0 = \iota$ and define $f_n, \iota_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ by the pullback diagram

$$\begin{array}{ccc} \mathcal{M}_{n+1} & \xrightarrow{\iota_n} & \mathcal{M}_n \\ \downarrow f_n & & \downarrow f_{n-1} \\ \mathcal{M}_n & \xrightarrow{\iota_{n-1}} & \mathcal{M}_{n-1}. \end{array}$$

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Then the nth iteration $\mathcal{F}^n$ of the topological automaton $\mathcal{F}$ is the covering $f_0 \circ f_1 \circ \cdots \circ f_{n-1} : \mathcal{M}_n \longrightarrow \mathcal{M}$ together with the map $\iota_0 \circ \iota_1 \circ \cdots \circ \iota_{n-1} : \mathcal{M}_n \longrightarrow \mathcal{M}$.

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If $\mathcal{F}$ is the dual Moore diagram of an invertible automaton $\mathcal{A}$, then $\mathcal{F}^n$ is the dual Moore diagram of the automaton describing the action of $\mathcal{A}$ on strings of length n.

$$
\begin{tikzcd}
 & \vdots \arrow[d, "f_4"] & \vdots \arrow[d, "f_3"] & \vdots \arrow[d, "f_2"] \\
\cdots \arrow[r, "\iota_4"] & \mathcal{M}_4 \arrow[r, "\iota_3"] \arrow[d, "f_3"] & \mathcal{M}_3 \arrow[r, "\iota_2"] \arrow[d, "f_2"] & \mathcal{M}_2 \arrow[d, "f_1"] \\
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We get three inverse limits $\lim_f \mathcal{F}$, $\lim_\iota \mathcal{F}$ and $\lim_{f,\iota} \mathcal{F}$ with self-maps ι_∞, f_∞ and Δ.

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Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a topological automaton such that $\mathcal{M}$ is a compact path connected and locally path connected (orbi)space. $\mathcal{F}$ is *contracting* if there exists a length structure on $\mathcal{M}$ and $\lambda < 1$ such that for every rectifiable path γ in $\mathcal{M}_1$

$$\mathrm{length}(\iota(\gamma)) \leq \lambda \cdot \mathrm{length}(\gamma),$$

where length of γ is computed with respect the lift of the length structure by f.

Theorem

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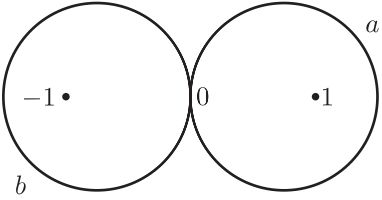
Hence it is a covering of $\mathbb{C} \setminus \{\pm 1\}$ by the subset $\mathbb{C} \setminus f^{-1}(\{\pm 1\}) = \mathbb{C} \setminus \{\pm 1, \pm 2\}$. The fundamental group is generated by



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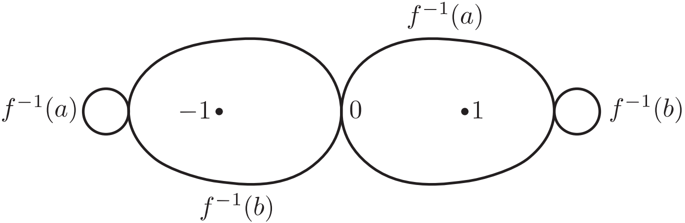

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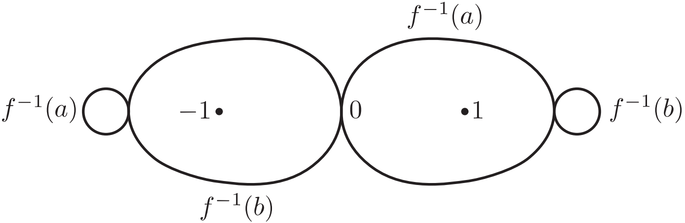

We get

$$\iota_*(a^2) = a, \quad \iota_*(b^2) = b, \quad \iota_*(a^b) = 1, \quad \iota_*(b^a) = 1.$$

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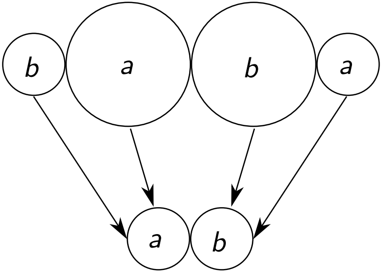

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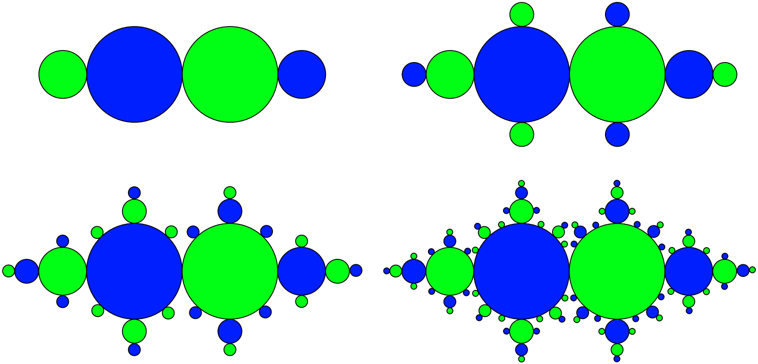

Wir gehen von zwei gleichseitigen Dreiecken $\triangle A_1 A_2 A_3$ und $\triangle A_1 A_4 A_5$ mit der Seite a aus, die an der Ecke A_1 aneinanderstoßen (Fig. 2). Sie bilden zusammen den geschlossenen polygonalen Zug $p_1 = A_1 A_2 A_3 A_4 A_5$, der die Ebene in 3 Bereiche teilt:

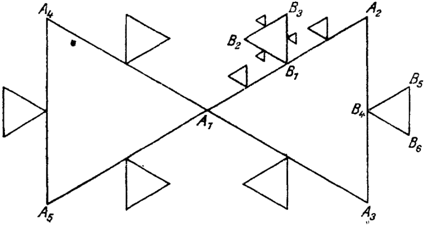

Fig. 2.

1. Das Innere von $\triangle A_1 A_2 A_3$: $\mathfrak{B}_1$.

2. Das Innere von $\triangle A_1 A_4 A_5$: $\mathfrak{B}_1'$.

3. Den Bereich $\mathfrak{B}_1''$, der den unendlich fernen Punkt enthält und vom ganzen polygonalen Zug p_1 begrenzt wird.

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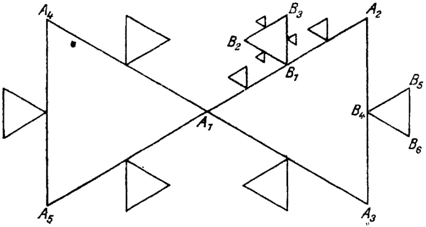

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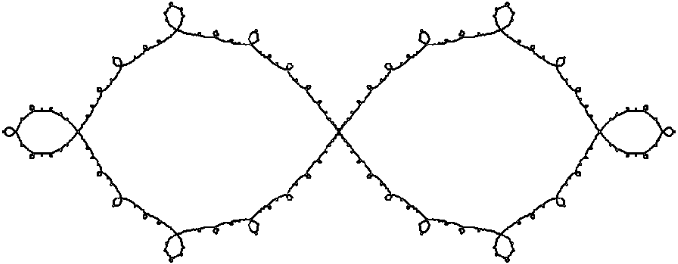
The original picture appears in a paper of Gaston Julia in 1918.

The Julia set of $-\frac{z^3}{2} + \frac{3z}{2}$



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for all $g \in \mathrm{Dom}\,\phi^N$.
A *model* of (G, ϕ) is a length space $\mathcal{X}$ on which G acts by isometries, properly and co-compactly and a contracting map $\Phi : \mathcal{X} \longrightarrow \mathcal{X}$ such that

$$\Phi(\xi \cdot g) = \Phi(\xi) \cdot \phi(g)$$

for all $g \in \mathrm{Dom}\,\phi$ and $\xi \in \mathcal{X}$.

If $(\mathcal{X}, \Phi)$ is a model of (G, ϕ), then we get a contracting topological automaton $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$, where $\mathcal{M} = \mathcal{X}/G$, If $(\mathcal{X}, \Phi)$ is a model of (G, ϕ), then we get a contracting topological automaton $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$, where $\mathcal{M} = \mathcal{X}/G$, $\mathcal{M}_1 = \mathcal{X}/G_1$, If $(\mathcal{X}, \Phi)$ is a model of (G, ϕ), then we get a contracting topological automaton $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$, where $\mathcal{M} = \mathcal{X}/G$, $\mathcal{M}_1 = \mathcal{X}/G_1$, $f : \mathcal{X}/G_1 \longrightarrow \mathcal{X}/G$ is induced by the identity map,

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Then the iterated monodromy group of $\mathcal{F}$ is $(G/N_\phi, \phi/N_\phi)$, where $N_\phi = \bigcup_{n\geq 1, g\in G} g^{-1} \cdot \mathrm{Dom}\,\phi^n \cdot g$.

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The inverse limit of the spaces $\mathcal{X}/\mathrm{Dom}\,\phi^n$ with respect to the maps induced by Φ depends only on (G, ϕ) and is called the *limit space* of (G, ϕ).

Rips complexes

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$$\Phi(g) := \phi(r^{-1}g), \qquad r \in R, \quad r^{-1}g \in \mathrm{Dom}\,\phi$$

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A natural candidate is a *Rips complex* of G. If $S = S^{-1} \ni 1$ is a generating set, then define $\Gamma(G, S)$ to be the simplicial complex with vertex set G in which $A \subset G$ is a simplex iff $g^{-1}A \subset S$ for every $g \in A$.

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Cut-and-paste rules

Baricentric subdivision of $\Gamma(G, S)$ coincides with the geometric realization of the poset of the sets of the form $A \cdot g$ for $A \subset S$ and $g \in G$.

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The complexes $\mathcal{M}_n$ are obtained by taking d^n copies of T and pasting them together by copies of κ_h according to a simple recursive rule.

Theorem

There exists a generating set S of G and a number n such that $\Phi^n : \Gamma(G, S) \longrightarrow \Gamma(G, S)$ is homotopic through maps Ψ satisfying $\Psi(\xi \cdot g) = \Psi(\xi) \cdot \phi^n(g)$ to a contracting map.

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In this way we get a model of the virtual endomorphism (G, ϕ^n), which is good enough to get combinatorial approximations of the Julia sets.

A more explicit version of the theorem is algorithmic. There is an algorithm which, given the iterated monodromy group of an expanding dynamical system, produces the complex T and the pasting rules κ_h, thus giving a recurrent description of the complexes $\mathcal{M}_n$ approximating the Julia set.