The geometry of spheres in free abelian groups

Moon Duchin
University of Michigan

joint work with Samuel Lelièvre and Christopher Mooney

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Introduction

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We will study sphere-averages:

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{x \in S_n} \frac{1}{n} f(x).$$

Cone measure

Given finite genset $(\mathbb{Z}^d, S)$,

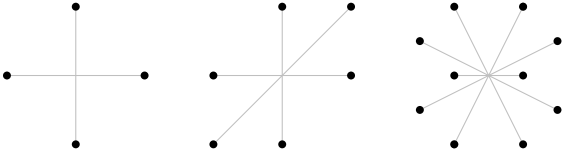

Figure: Some gensets: $S_{\text{std}} = \pm\{e_1, e_2\}$, $S_{\text{hex}} = \pm\{e_1, e_2, e_1 + e_2\}$, $S_{\text{chess}} = \{(\pm 2, \pm 1), (\pm 1, \pm 2)\}$ (with irrelevant generator thrown in).

Cone measure

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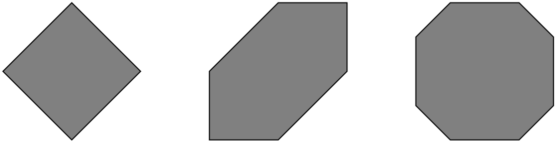

Figure: Convex hulls Q.

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Given finite genset $(\mathbb{Z}^d, S)$, let Q be the convex hull of S in $\mathbb{R}^d$, let $L = \partial Q$,

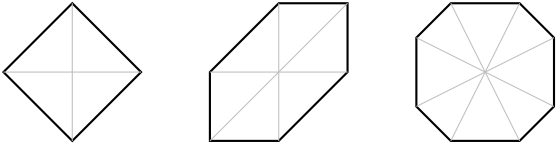

Figure: Boundary polyhedra L.

Cone measure

Given finite genset $(\mathbb{Z}^d, S)$, let Q be the convex hull of S in $\mathbb{R}^d$, let $L = \partial Q$, and let $\hat{A}$ be the cone from $A \subseteq L$ to $\mathbf{0}$, so that $Q = \hat{L}$.

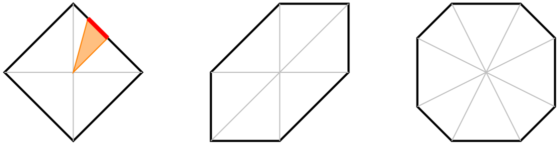

Figure: A cone.

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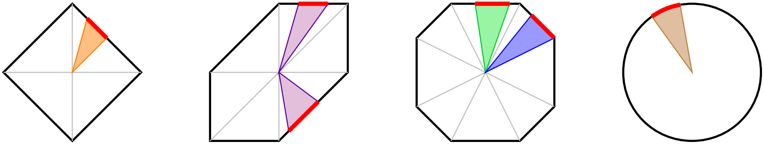
Given finite genset $(\mathbb{Z}^d, S)$, let Q be the convex hull of S in $\mathbb{R}^d$, let $L = \partial Q$, and let $\hat{A}$ be the cone from $A \subseteq L$ to $\mathbf{0}$, so that $Q = \hat{L}$.

Define the *cone measure* by $\mu(A) = \mu_L(A) = \frac{\mathrm{Vol}(\hat{A})}{\mathrm{Vol}(Q)}$.



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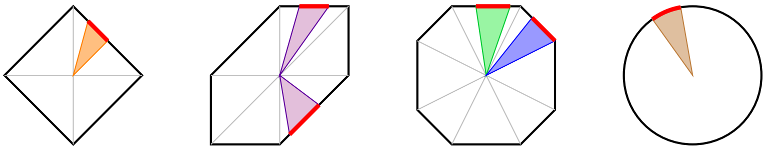

Figure: Cone measure.

As we will discuss below, $\frac{1}{n} S_n \to L$ as a Gromov-Hausdorff limit. We show *counting measure on spheres converges to cone measure on L.*

Limit shape, limit measure

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Theorem (Limit shape and limit measure)

For any finite presentation $(\mathbb{Z}^d, S)$ and any function $f : \mathbb{Z}^d \to \mathbb{R}$ asymptotic to a homogeneous $g : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{x \in S_n} \frac{1}{n} f(x) = \int_L g(x)\, d\mu(x).$$

So group averaging problem reduces to a problem in convex geometry.

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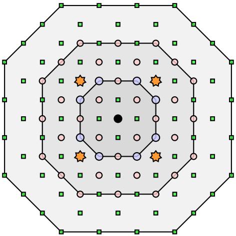

Figure: The *chess-knight metric*.

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- The annular region $\Delta_n L := nQ \setminus (n-1)Q$ is covered by $S_{n-1} + Q$.
- How much word-length is used to fill in Q? Let $K = \max|\mathbb{Z}^2 \cap Q|$.
  Then $|\mathrm{w}|$ and $\|\mathrm{w}\|_L$ differ by at most K. (Burago 1992)

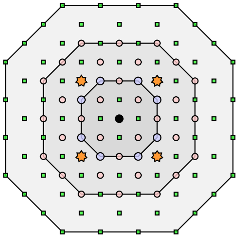
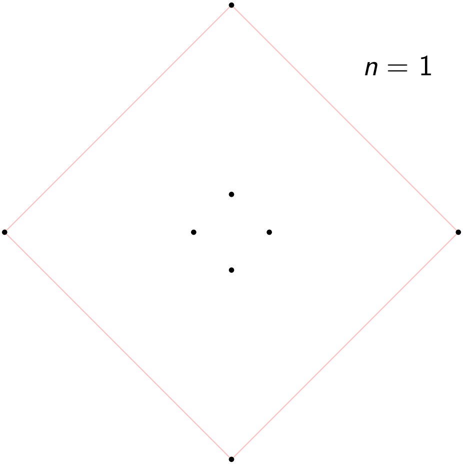


Figure: The *chess-knight metric*.

Convergence to the limit shape

$\mathbb{Z}^2$ with $S = \{ 6e_1, e_1, 6e_2, e_2 \}$. Watch $\frac{1}{n} S_n$ converge to L.



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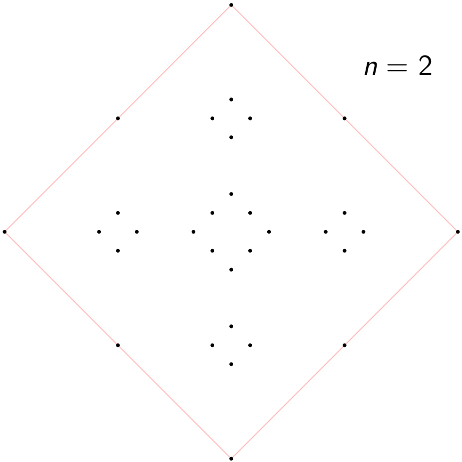

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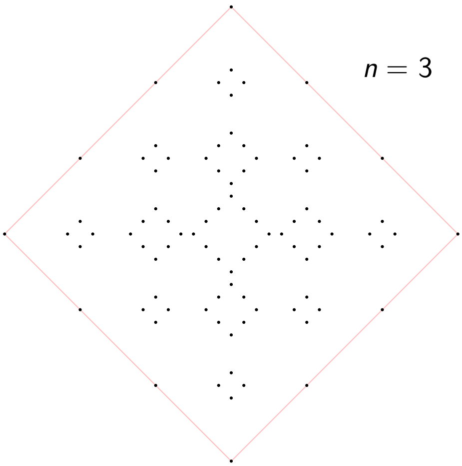

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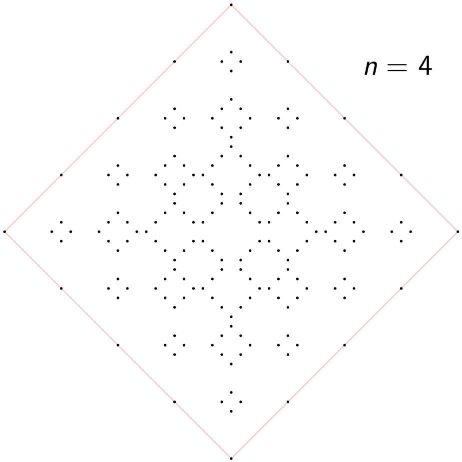

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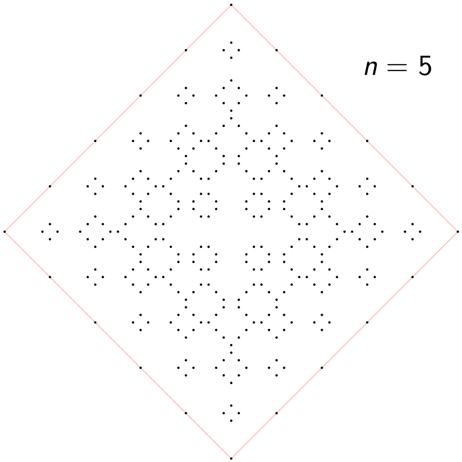

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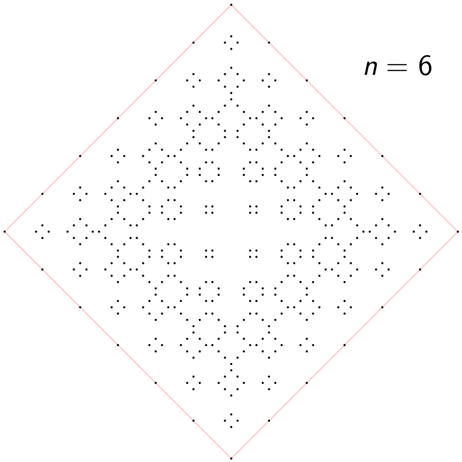

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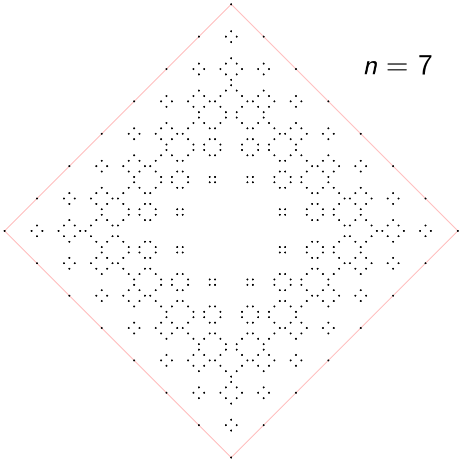

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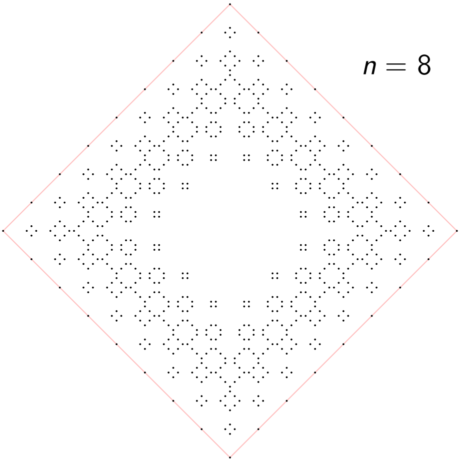

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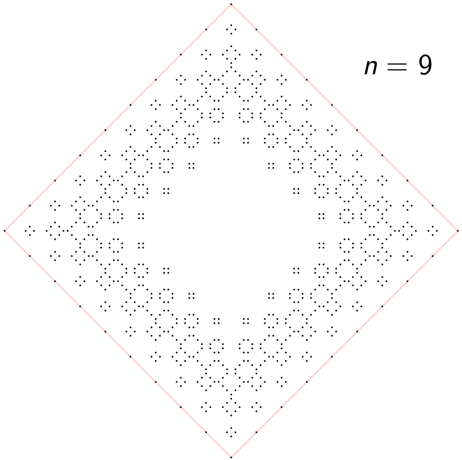

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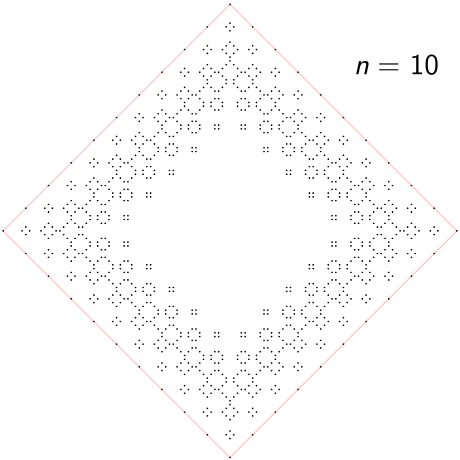

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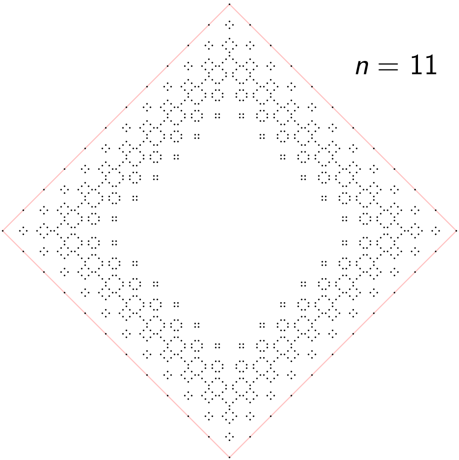

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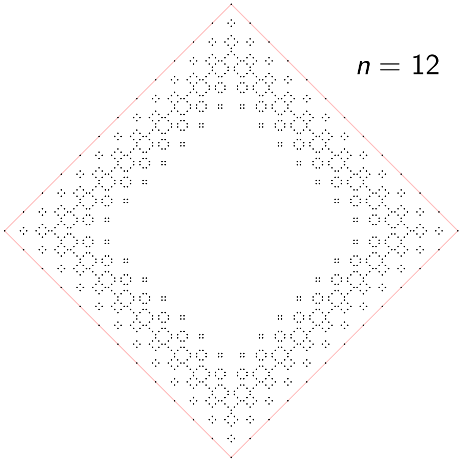

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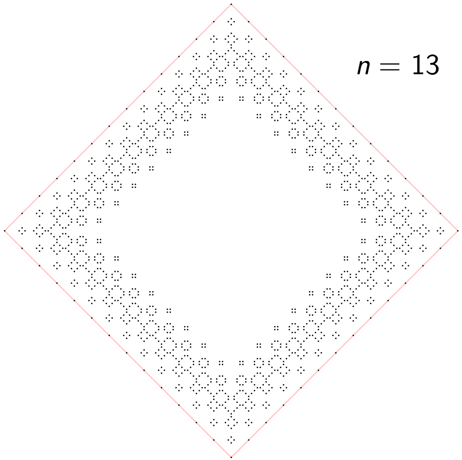

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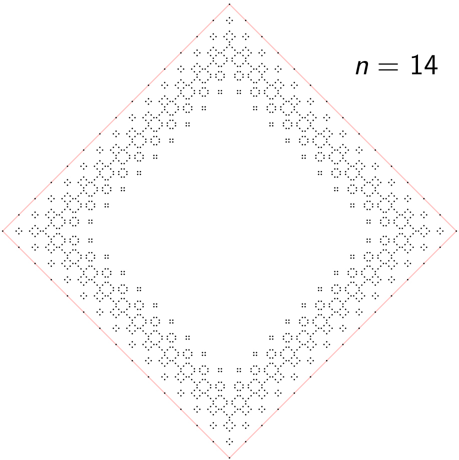

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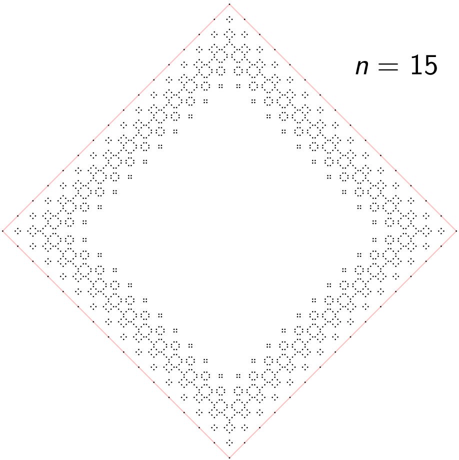

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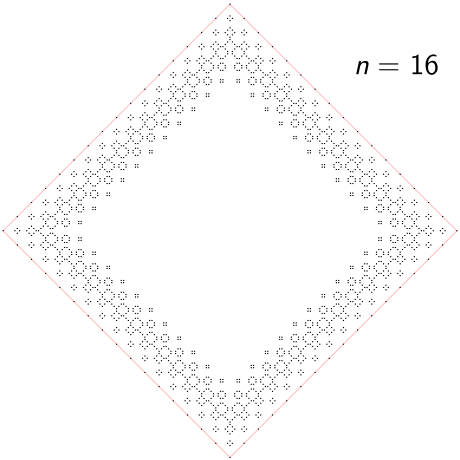

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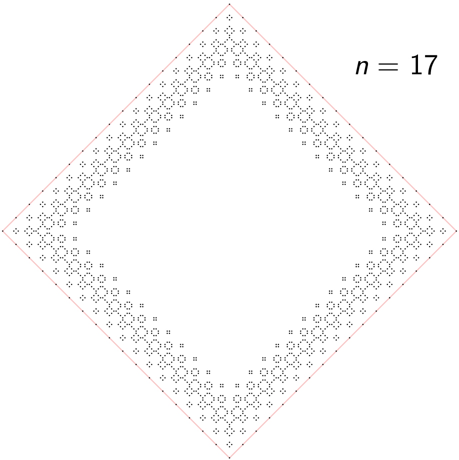

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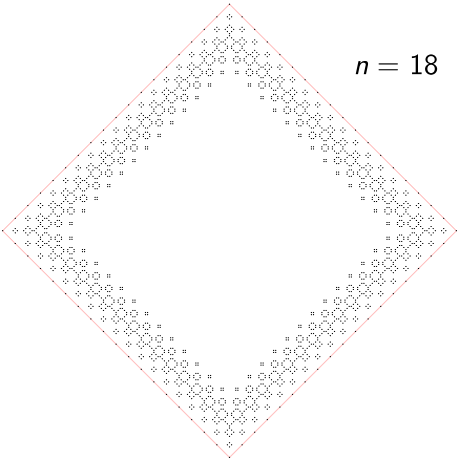

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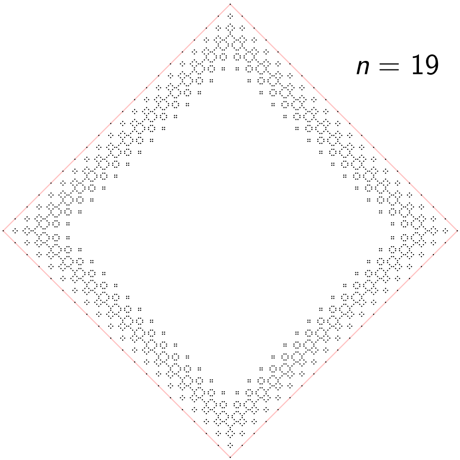

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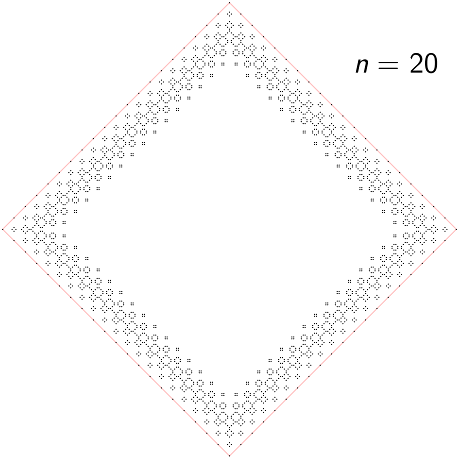

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Application: Density

Theorem (Density)

For any asymptotically homogeneous function $f : \mathbb{Z}^d \to \mathbb{R}$,

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Notably different from scale-invariant functions on $\mathbb{Z}^d$, or from any functions on hyperbolic groups, where ball-average equals sphere-average.

Introducing *sprawl*

Question: what is the average distance between two points in a large sphere?

$$E(G,S) := \lim_{n\to\infty} \frac{1}{|S_n|^2} \sum_{x,y\in S_n} \frac{1}{n} d(x,y).$$

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- $E(\mathbb{Z}^d, S)$ *depends on* S*.*
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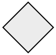
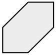
Note $E(L) = E(TL)$ for $L \in GL(d,\mathbb{R})$.

Sprawl in $\mathbb{Z}^2$

We developed the *cutline* algorithm for computing sprawls of polygons and used it to make calculations, showing the extent of dependence on S.



$E(P_4) = \frac{4}{3}$



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Sprawl in $\mathbb{Z}^2$

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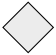

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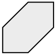

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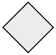

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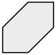

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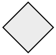

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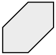

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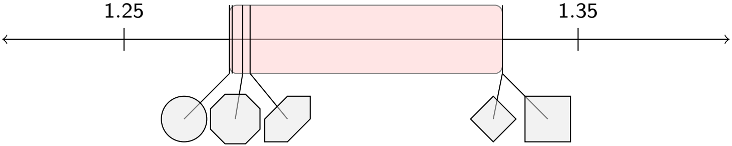

Figure: Ranges of sprawls.

Sprawls of hexagons and three-generator presentations

We know that $E(\Omega)$ can take all values in $[4/\pi, 4/3]$, which implies that $E(\mathbb{Z}^2, S)$ can take a dense set of values in that range. (Rational approximation.)

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Besides lots of empirical evidence, here is some good rigorous evidence.

Theorem

$$\{E(H) : \text{ hexagons } H\} = [23/18, 4/3]\,.$$

Thus, $\frac{23}{18} \leq E(\mathbb{Z}^2, S) \leq \frac{4}{3}$ *whenever* $|S| \leq 6$.

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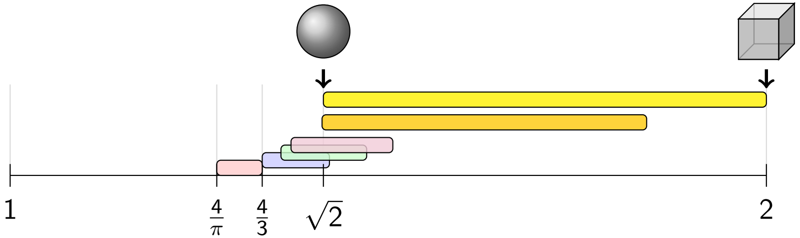

Figure: Ranges of sprawls: $d = 2, 3, 4, 5, 100, \infty$.

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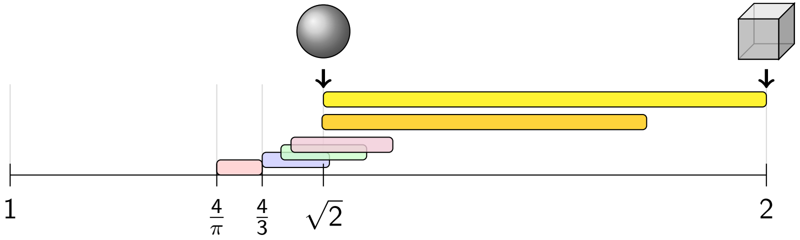

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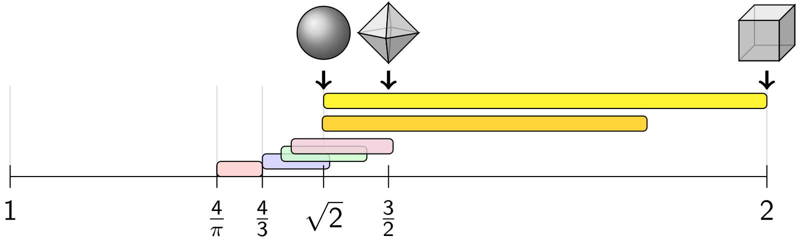

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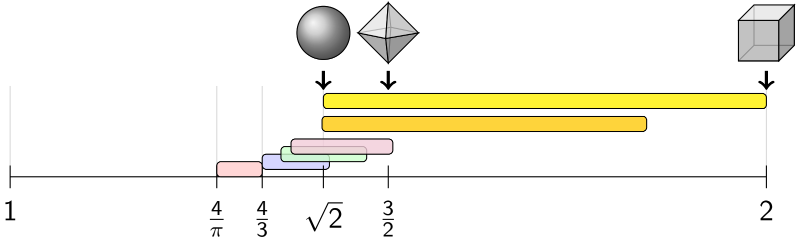

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Conclusion: to make a free abelian group look as hyperbolic as possible, use the nonstandard generators $S_{\text{cube}} = \{\pm e_1 \pm e_2 \cdots \pm e_d\}$.