On the dimension growth of groups

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The growth rate of $k(\lambda)$ is a q.i. invariant.

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For many groups, the dimension growth is constant: hyperbolic, toric relatively hyperbolic, RAAGs, the mapping class group, nilpotent groups, etc.

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$\mathbb{Z} \wr \mathbb{Z}$, the Grigorchuk group, the R. Thompson group F, etc. have infinite asymptotic dimension and the question about dimension growth is natural for these groups.

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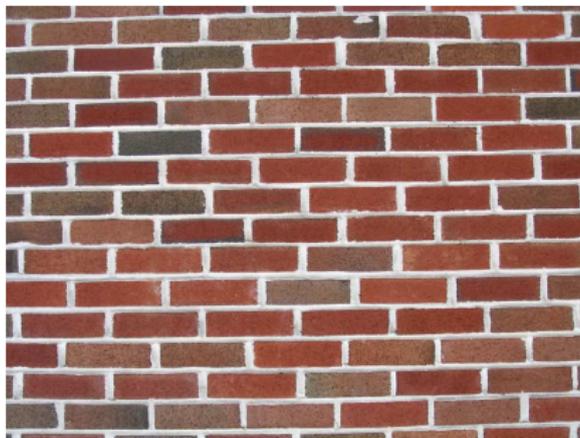$D(\lambda)$ is linear, and $k(\lambda)$ is constant, we get the Assouad-Nagata dimension.

We do not know any finitely generated group where more than exponential control is needed.

How large should the brick be?

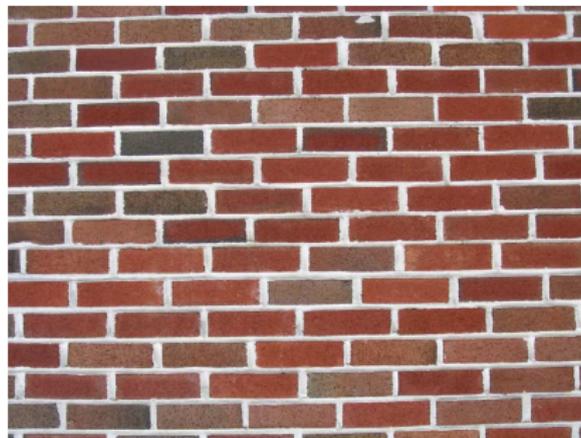The Assouad-Nagata dimension of $\mathbb{Z}^n$ is n

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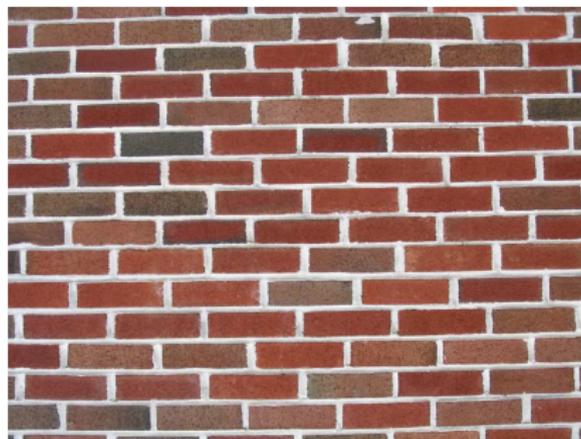
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Problem. What is the smallest size of a brick? What if we color in a different way (not by bricks)?

Connection with quasi-isometry and coarse embeddings

A map of metric spaces $\phi : X \to Y$ is called a *coarse embedding* if there are strictly monotone tending to infinity functions $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ and a number $r > 0$ such that

$$\rho_1(d_X(x, x')) \leq d_Y(\phi(x), \phi(x')) \leq \rho_2(d_X(x, x'))$$

for all $x, x' \in X$ with $d(x, x') \geq r$.

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Let $\phi : X \to Y$ be a coarse embedding with functions ρ_1, ρ_2: Then

$$(\lambda, D)\text{-dim}(Y) \geq (\rho_2^{-1}(\lambda), \rho_1^{-1}(D))\text{-dim}(X).$$

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Corollary. The dimension growth of any finitely generated group is at most exponential (with constant, hence any, control).

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Hence Gromov random groups containing expanders have exponential asymptotic dimension growth.

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Then the dimension growth of G is exponential.
Proof. Let $V_G = \cup_{i=1}^{k+1} U_i$ be a coloring of the vertices of G in $k + 1$ colors such that all λ-clusters U_i^j have diameters at most d. Take $r > d + \lambda$ and consider the graph $G_r = (V_r, E_r)$. Let $W_i^j = U_i^j \cap G_r$. We have $\cup W_i^j$ equal to the set V_r of all vertices of G_r. Note that $N_{\lambda/2}(W_i^j)$ has at least $(1 + \varepsilon)^{\lambda/2}$ elements. Since different λ-clusters of the same color are λ-disjoint, we have that the sum of $|N_{\lambda/2}(W_i^j)|$ is at most $|V_r|(k + 1)$. On the other hand, that sum is at least $(1 + \varepsilon)^{\lambda/2}$ times the sum of cardinalities $|W_i^j|$, i.e. at least $|V_r|(1 + \varepsilon)^{\lambda/2}$. Hence $k + 1 \geq (1 + \varepsilon)^{\lambda/2}$.

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Proof. Every finite subset M of $\mathbb{N}$ corresponds to a vector $v(M)$ from $\mathbb{Z}^\infty$ with coordinates 0, 1 in the natural way. Choose any $k \geq 1$. Let $P_k(\mathbb{N})$ denote the set of all k-element subsets of $\mathbb{N}$. Every finite coloring of $\mathbb{Z}^\infty$ induces a finite coloring of $P_k(\mathbb{N})$. By Ramsey there exists a subset $M \subseteq \mathbb{N}$ of size $2k$ such that all k-element subsets of M have the same color. Therefore we can find subsets $T_1, T_2, \ldots, T_k$ of size k from M such that the symmetric distance between T_i and T_{i+1} is 2, $i = 1, \ldots, k-1$, and T_1, T_k are disjoint. Then the vectors $v(T_1), \ldots, v(T_k)$ from $\mathbb{Z}^\infty$ form a monochromatic 2-path of diameter $\geq 2k$.

The dimension growth of $\mathbb{Z}^n$.

Let G be the binary cube $\{0,1\}^n$ with the ℓ_1-metric. Then for every $r > 0$, such that $\varepsilon = \frac{n}{r+1} - 2 > 0$, G satisfies property $(P_r(\epsilon))$.

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[**The controlled 4-dimension of a binary n-cube**] The binary n-cube $\{0,1\}^n$, $n > 64$, cannot be colored by n colors such that each 4-cluster of every color has diameter less than $\leq \sqrt{n}/4$, i.e. $(4, \sqrt{n}/4)$-dim $(\mathbb{Z}^n) = n$ for $n > 64$.

The main open problem.

Problem. Is it true that for some $\alpha > 0$, $k_{\mathbb{Z}^n}(\lambda) = O(n^\alpha)$ for every λ.

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We do not know the answer for $\lambda = 2, \alpha = 1$. We also do not know whether $k_{\mathbb{Z}^n}(\lambda)$ is bounded for every λ as a function of n.

Connection with the game of Hex

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Theorem. There is always a winner in the game of Hex. Hence if we color $\mathbb{Z}^n$ with ℓ_∞-metric in n colors there is always arbitrary long monochromatic paths. Thus $1\text{-dim}(\mathbb{Z}^n, \ell_\infty) = n$ for every n.

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Remark. The game of Hex on the plane corresponds to the hexagonal tessellation of the plane and the graph metric on its dual graph. As we know from percolation theory (Smirnov), hexagonal lattice is much easier than square lattice.
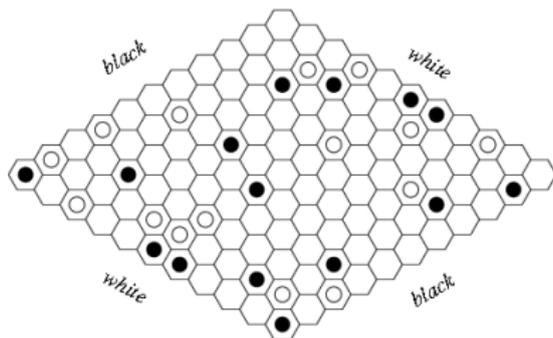
The dimension growth of the R. Thompson group F.

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Hence the dimension growth of F with some exponential control is exponential.

What is the dimension growth of F? Is super-exponential control required?

Upper bounds. Connection with the Kolmogorov-Ostrand dimension

We say that Kolmogorov-Ostrand dimension of X is $\leq n$ if for every $m \geq 0$ there exists a coloring of X in $m + n$ colors (every point may be colored in many colors) such that the diameters of all λ-clusters are uniformly bounded.

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For Assouad-Nagata dimension it was proved by Brodskiy, Dydak, Levin, and Mitra.

Proof. Suppose $KO - dim(X) = n_1, KO - dim(Y) = n_2$. Consider colorings of X and Y in $n_1 + n_2 + m$ colors (as required by the definition). Then color (x, y) in color i if both x and y has color i. This gives a required coloring of $X \times Y$.

The dimension growth of solvable subgroups of F

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Problem. What is the dimension growth of $\mathbb{Z} \wr \mathbb{Z}$?