The relative rank gradient and the subgroup structure of certain amenable groups

R. Grigorchuk

Texas A & M University

i don't remember having met Bob Gilman,
But i have often seen his name as author of
interesting papers.

Pierre de la Harpe

Happy Birthday Bob!

(I). Relative rank gradient

G - f.g. group

$G > H_1 > H_2 > \cdots > H_n > \cdots$ - strictly descending chain of

subgroups of finite index (only such will be used in the talk)

Schreier inequality

$G > H \qquad [G:H] < \infty$

$$\boxed{[d(H)-1] \leq [d(G)-1][G:H]}$$

$d(G)$ = minimal # of generators of G

$$RG(G) = \inf_{\{H_n\}} RG(G, \{H_n\})$$

limit exists

$$RG(G, \{H_n\}) = \lim_{n\to\infty} \frac{d(H_n)-1}{[G:H_n]}$$ – rank gradient

Th. [Abert, Jaikin-Zapirain, Nikolov]. G amenable

$H_n \triangleleft G$ (normal chain), $\bigcap_{n=1}^{\infty} H_n = \{1\}$

$\Rightarrow \boxed{RG = 0}$.

Q_1 [A J-Z N] Let G be a f.g. amenable group. Is it true that $RG(G, \{H_n\}) = 0$ for any chain with trivial intersection?

Def. 1) if $H < G$ Core(H) = maximal normal in G subgroup of H.

2) $\mathrm{core}(\{H_n\}) = \mathrm{core}\left(\bigcap_n H_n\right)$.

Conjecture. G is a f.g and amenable then $RG = 0$ for any chain with trivial core.

p - prime

$k \geq 1$

$$\boxed{\mathcal{L}_{k,p} = \left(\mathbb{Z}/p\mathbb{Z}\right)^k \wr \mathbb{Z}}$$

$$= \left(\bigoplus_{\mathbb{Z}} \left(\mathbb{Z}/p\mathbb{Z}\right)^k \right) \rtimes \mathbb{Z}$$

$p = 2$, $k = 1$ $\mathcal{L} = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ - lamplighter

Th [Gr. Kravchenko] $\forall\, k \geq 1$ and prime p the above conjecture is correct.

---

if $RG = 0$ then consider

$$rg(n) = rg_{(G, \{H_n\})}(n) = \frac{d(H_n) - 1}{[G : H_n]}$$

- relative rank gradient function.

Q_2 What can be rate of decay of $rg(n)$ when $n \to \infty$?

Can be arbitrary fast for scale-invariant groups.

II Scale-invariance. $G, \{H_n\}$ as before

Def. a) G is $\overbrace{\text{weakly scale-invariant}}^{\text{wsi}}$ if there is a chain $\{H_n\}$ with $H_n \simeq G$, $n = 1, 2, \dots$

Remark. G is wsi if and only if G contains a proper subgroup $H < G$ isomorphic to G.

b) G is scale-invariant of type si_1, si_2, si_3 if there is a chain $\{H_n\}$ as above and

si_1 $\bigcap_n H_n = \{1\}$

si_2 $\bigcap_n H_n = \text{finite}$

si_3 $\mathrm{Core}(\bigcap_n H_n) = \{1\}$

$si_1 \Rightarrow si_2$

$\Rightarrow si_3$

BSi - Benjamini s.i.

c) G is <u>strongly</u> scale invariant of type ssi_1, ssi_2, ssi_3 if it is scale invariant of corresponding type si_1, si_2, si_3 and additionally $H_n = \varphi^n(G)$ where $\varphi: G \to G$ is injective endomorphism with $\varphi(G)$ a proper subgroup of G. [Nekrashevych, Pete].

Conjecture [Benjamini, 2006]. if a f.g. group G is Bsi then G is virtually nilpotent.

incorrect. Grig. Zuk 2002 implicitely

Nekrashevych, Pete 2011 (in fact 2007) explicitely

in fact

Lamplighter, BS(1, m) - Baumslag-Soliter

$m > 1$ are scale invariant

Th. [Gr Kr] $\mathcal{L}_{k,p}$ is si, (and hence BSi) for $\forall k, p$.

The proof is based on self-similar (automaton) presentation of $\mathcal{L}_{k,p}$ and on essential freenes of the action on the boundary of rooted tree (the topics to be discussed later).

---

if G is scale invariant w.r. to chain $\{H_n\}$, $H_n \simeq G$ $\forall n$

then

$$rg_{(G, \{H_n\})} = \frac{d(G)-1}{[G:H_n]}$$

$\Rightarrow$ decay can be made arbitrary fact (by delition of some members of the chain)

Reasonable to consider subnormal p-chains i.e.

$\forall n \qquad H_{n+1} \triangleleft H_n \qquad [H_n : H_{n+1}] = p$

Th. Any subgroup of index p of $\mathcal{L}_{n,p}$ is isomorphic either to $\mathcal{L}_{n,p}$ or to $\mathcal{L}_{pn,p}$ (and both cases occure).

Th. Suppose that $g(i): \mathbb{N} \to \mathbb{N}$ is such that $g(0) = p+1$ and for each i

$$g(i+1) = \begin{cases} g(i) & \text{or} \\ pg(i) - p + 1 \end{cases}$$

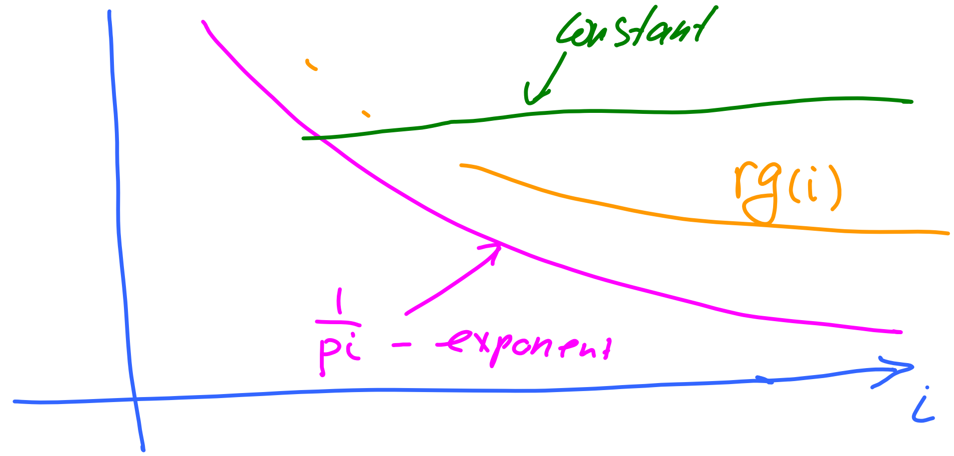
and suppose that the set $\{i \mid g(i) = g(i+1)\}$ is infinite.

Then there is a descending subnormal p-chain $\{H_i\}$

of subgroups of $\mathbb{Z}_{1,p}$ such that $H_0 = \mathbb{Z}_{1,p}$, $d(H_i) = g(i)$ and $\bigcap_i H_i = \{1\}$.

Corollary. There is $2^{\aleph_0}$ different types of decay of the relative rank gradient of subnormal p-chains in $\mathbb{Z}_{1,p}$ with trivial intersection.

$$rg(i+1) = \begin{cases} \frac{rg(i)}{p} & \text{or} \\ rg(i) + \frac{1}{p^{i+1}} & \end{cases}$$



Def. 1) Abert, Nikolov.

$$e_G(n) := \min_H \left\{ \frac{d(H)}{[G:H]} \;\middle|\; [G:H]=n \right\}$$

2)

$$e^*_G(n) := \min_H \left\{ \frac{d(H)}{[G:H]} \;\middle|\; H \triangleleft G,\ [G:H]=n \right\}$$

Th. Ab. Nik. Let G be a d-generated metabilian group. There is a constant c such that for all large $n \in \mathbb{N}$

$$e_G(n) \le \frac{c \log n}{n^{\frac{1}{d+2}}}$$

fastest possible decay of $e_G(n)$

Remark. G scale invariant $\Longrightarrow$ $e_G(n) \le \dfrac{c}{n}$ (at least for $n = q^i$, $i = 1, 2, \ldots$, where $[G:H] = q$, $H \simeq G$.)

Th. Gr. Kr. For $\mathcal{L}_{1,p}$ and for $\mathcal{G} = \langle a, b, c, d \rangle$ (group of intermediate growth)

$$(*) \qquad \boxed{e^*(n) \sim \frac{\log n}{n}}$$

$$\mathcal{G} = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, \sigma^k((ad)^4), \sigma^k((adacac)^4), k \geq 0 \rangle$$

$$\sigma : \begin{cases} a \to aca & c \to b \\ b \to d & d \to c \end{cases}$$

I. LYSENOK presentation.

$\mathcal{L}_{n,p} \in EG$, — elementary amenable

$\mathcal{G} \in \underbrace{AG \setminus EG}_{\text{non elementary amenable}}$

Q. is it correct that $(*)$ hold for arbitrary f.g residually finite group?

$\mathcal{L}_{1,p}$	$\mathcal{G}$
1. Scale invariant (si_1)	1. Not scale invariant in any sense (non trivial Nagnibeda Gr)
2. All maximal subgroup have finite index	2. — // — E. Pervova
3. LERF	3. LERF (J. Wilson, Gr)
4. All weakly maximal subgroups are closed in the profinite topology	4. — // — (J. Wilson, Gr)
5. Complete description of weakly maximal subgroup	5. Partial results in this direction

6. Description of the lattice of normal subgroups

7. $\mathcal{L}_{1,p}$ is a self-similar group. Moreover $\mathcal{I}_{1,p}$ is a group generated by automaton of Mealy type with p states over alphabet with p symbols [(p,p)-type automaton]

8. The action of $\mathcal{L}_{1,p}$ on the binary rooted tree induced by automaton presentation has CSP (congruence subgroup property)

6. Partial results in this direction. $\mathcal{J}$ is just-infinite

8. $\mathcal{J} = G(\mathcal{A})$

$\mathcal{A}$ is (2,5) automaton 5-states

8. —//—

9. The above action induces essentially free action on the boundary ∂T of the tree

10. finite commutator width

9. — " — totally nonfree

10. Finite commutator width (≤ 20)

I. Lysenok, A. Miasnikov

IV Actions on rooted trees and automaton presentation.

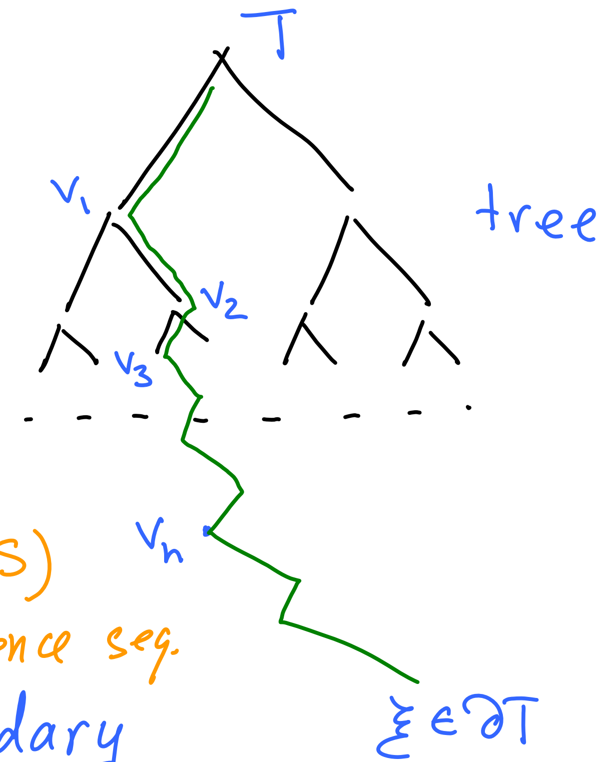
$G \curvearrowright T$

$St_G(v)$ - stabilizer of vertex v

$St_G(n)$ - stabilizer of level n (PCS)

principal congruence seq.

$\xi = \{v_n\}_{n=1}^{\infty}$ - point of the boundary



$H = St_G(\xi)$, $\quad H_n = St_G(\xi_n)$ $\quad \bigcap_{n=1}^{\infty} H_n = H$

Correspondence between descending chains of subgroups of finite index and actions on rooted trees

$$\{H_n\} \longleftrightarrow T = (V, E) \quad V = \{gH_n : g \in G, n \geq 1\}$$
$$E = \{(gH_n, hH_{n+1}) : hH_{n+1} \subset gH_n\}$$

the action on T is given by left multiplication.

$G \curvearrowright T \Rightarrow G \curvearrowright \partial T$ - Boundary

$(G, \partial T)$ - topological dynamics $\quad (G, \partial T, \mu)$ - metric DS

μ - uniform measure

action can be topologically free ($\forall g \in G$, $g \neq 1$ $\mathrm{Fix}(g)$ is meager)

or essentially free ($\forall g \in G$, $g \neq 1$ $\mu(\mathrm{Fix}(g)) = 0$)

$\Leftrightarrow$ for a typical point $x \in \partial T$ $St_G(x) = \{1\}$.

trivial stabilizer

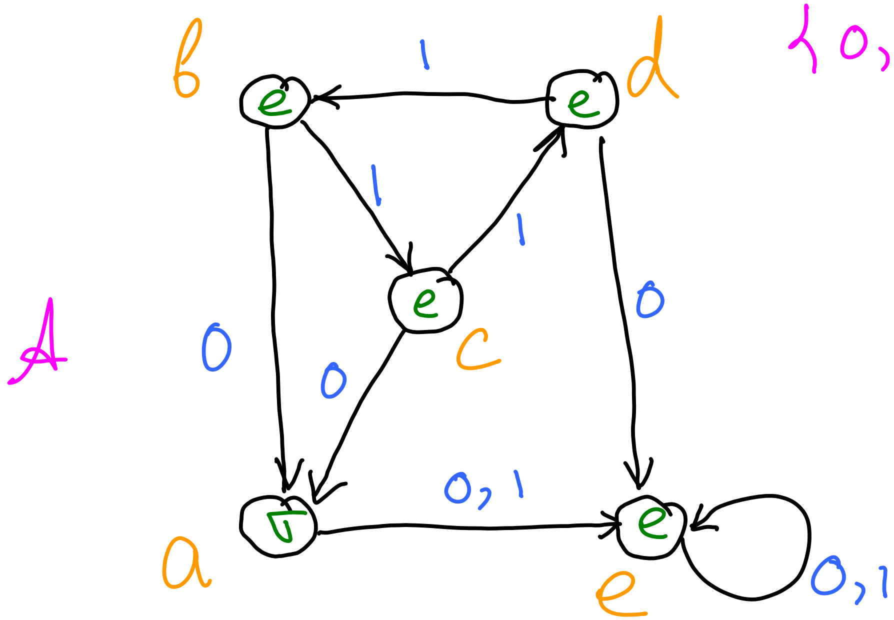
<u>Def.</u> $G \curvearrowright X$ is completely nonfree if

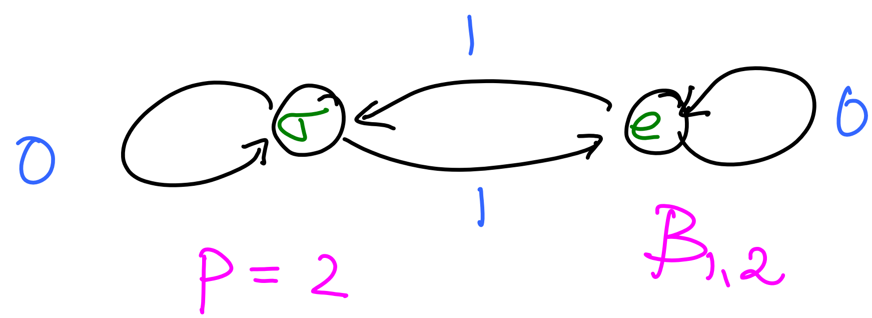
$$St_G(x) \neq St_G(y), \quad \forall x, y \in X, \quad x \neq y$$

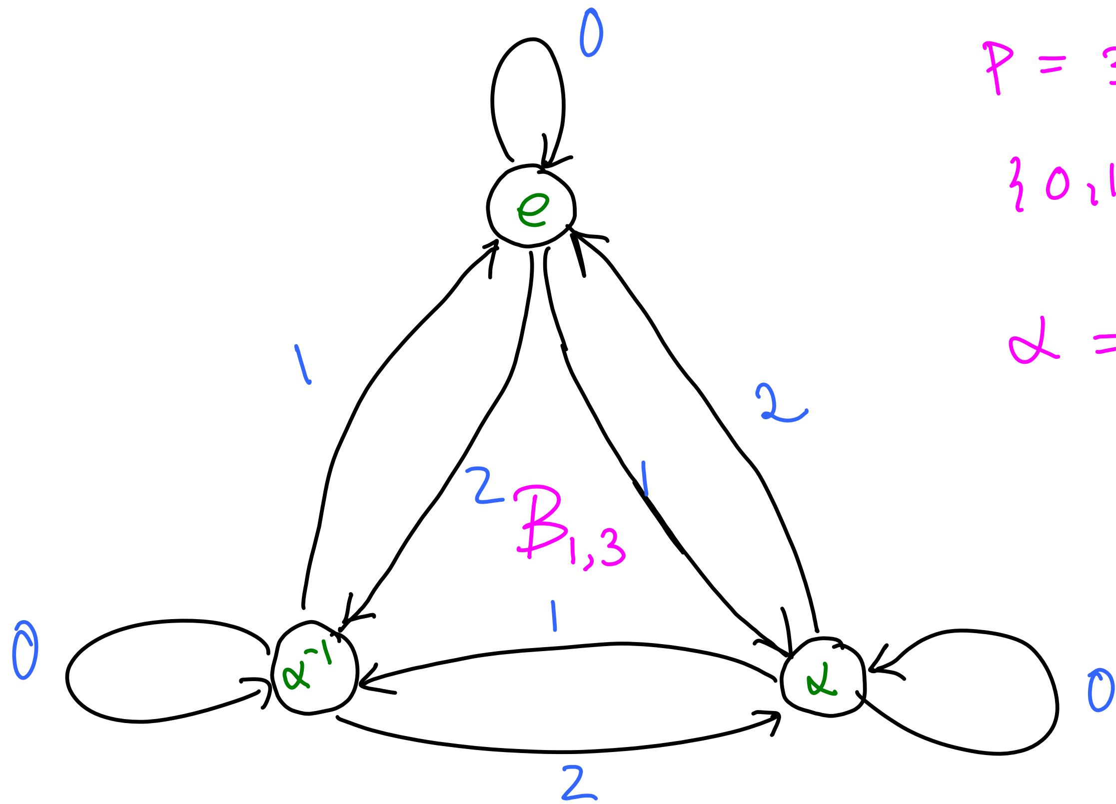
For actions defined by finite automata topological freenes is the same as essential freenes [Kambites, Silva, Steinberg]

Fact: $\mathcal{J}$ and $\mathcal{L}_{1,p}$ (p-prime) have automaton presentation: $\mathcal{J} \simeq G(\mathcal{A})$, $\mathcal{L}_{1,p} \simeq G(\mathcal{B}_p)$

$\{0,1\}$-alphabet, $Sym(2) = \{e, \sigma\}$







$p = 3$

$\{0, 1, 2\}$ – alphabet

$\alpha = (0\ 1\ 2)$ 3-cycle

$\mathcal{I}$ and $\mathcal{L} = \mathcal{L}_{1,2}$ act on binary rooted tree

$\mathcal{L}_{1,p}$ act on p-regular rooted tree. (and its boundary ∂T)

$\mathcal{L}_{2,2} = (\mathbb{Z}/2\mathbb{Z})^2 \wr \mathbb{Z}$ also has automaton presentation. Savchuk.

Self-replicating property + freeness $\Rightarrow$ scale invariance.

$\Rightarrow$ reasonable to search for self-similar groups acting freely on boundary of rooted tree.

D. Savchuk & G. Complete classification of (2, 3) automaton groups acting freely on a boundary of a tree.

Th. $\mathcal{X}_{1,p}$

1). The $\overset{\text{automaton}}{\text{action}}$ on T_p has CSP (congruence s.p) w.r. $\{St(n)\}$ and hence

$$\overline{\mathcal{X}_{1,p}} \simeq \hat{\mathcal{X}}_{1,p}$$

closure in AutT

profinite completion

2) $$rg_{(\mathcal{X}_{1,p},\{St(n)\})}(n) = \frac{p^e}{p^{e+n}}$$

$\mathcal{G} = \langle a,b,c,d \rangle$

1) — // — on T_2

— // —

$$\overline{\mathcal{G}} \simeq \hat{\mathcal{G}}$$

Corollary: We get self-similar (automaton) presentation of profinite completions.

$$rg_{(\mathcal{G},\{St(n)\})}(n) = \frac{7\cdot 2^{n-3}}{5\cdot 2^{n-3}+2}$$

2

where ℓ is smallest such that $m \le p^e$

3. $St_{\mathcal{L}_{1,p}}(v) \simeq \mathcal{L}_{1,p}$ $\forall$ vertex v

$$St_{\mathcal{L}_{1,p}}(\xi) \simeq \begin{cases} \mathbb{Z} & \text{if } \xi \in \partial T \text{ is eventually periodic} \\ \{1\} & \text{otherwise} \end{cases}$$

and hence action is essentially free.

3. The action of $\mathcal{G}$ on ∂T_2 is completely nonfree and $\forall\, \xi \in \partial T_2$

$St_{\mathcal{G}}(\xi)$

is weakly maximal subgroup

in fact: For any branch and even weakly branch group the action on ∂T is completely nonfree and $St(\xi)$ are weakly maximal

$\forall\ \xi = \{v_n\} \in \partial T_2$

$$rg_{(\mathcal{G}, \{St_{\mathcal{G}}(v_n)\})} = \frac{4n+4}{2^n}$$

Kravchenko Bartholdi

(V) Automaton presentation of $\mathcal{L}_{1,p}$ and iwasawa theory

T_p – p – regular rooted tree

$$\partial T_p \ni w = w_0 w_1 \dots \quad \overset{\alpha}{\longmapsto} \quad F_w(t) = \sum_{i=0}^{\infty} w_i t^i \in \mathbb{Z}_p[[t]]$$

$$w_i \in \mathbb{F}_p$$

Proposition. $\hat{\mathcal{L}}^{(p)}_{n,p} \simeq \left(\mathbb{F}_p[[\mathbb{Z}_p]]\right)^n \rtimes \mathbb{Z}_p$

completed group algebra over $\mathbb{F}_p$ of $\mathbb{Z}_p$.

p-adic integers

Embedding of $\mathbb{Z}_p$ into $\left(\mathbb{F}_p[[t]]\right)^*$

$$\mathbb{Z}_p \ni a = \sum_{i=0}^{\infty} a_i p^i \longrightarrow (1+t)^a = \prod_{i=0}^{\infty} \left(1+t^{p^i}\right)^{a_i}$$

$$\mathbb{F}_p[[\mathbb{Z}_p]] \simeq \mathbb{F}_p[[t]] \leftarrow \text{Iwasawa}$$

$$\begin{tikzcd}
\mathcal{L}_{1,p} \arrow[r, phantom, "\circlearrowright"] \arrow[dd, dashed] & \hat{\mathcal{L}}^{(p)}_{1,p} \arrow[d, phantom, "\cong" rotate=90] & \text{action by left multiplication} \\
 & \mathbb{F}_p[[\mathbb{Z}_p]] \rtimes \mathbb{Z}_p \arrow[d, "\pi_2"] & \\
\mathcal{L}_{1,p} \arrow[r, phantom, "\circlearrowright"] & \mathbb{F}_p[[\mathbb{Z}_p]] \simeq \mathbb{F}_p[[t]] \overset{\alpha}{\approx} \partial T_p &
\end{tikzcd}$$

The latter is automaton action of $\mathcal{L}_{1,p}$ on ∂T_p:

$\mathcal{L}_{1,p} = \langle a, b \rangle$

action of generators on formal power series

$$\begin{cases} a: & F(t) \to F(t) + 1 \\ b: & F(t) \to (1+t)\,F(t), \end{cases} \qquad F(t) \in \mathbb{F}_p[[t]]$$

VI Invariant random subgroups

G - countable group

$\mathrm{Sub}(G) \subset \{0,1\}^G$ Tychonoff topology

$G \geq H \rightsquigarrow \chi_H \in \{0,1\}^G$ - characteristic function

$G \curvearrowright G$ $\curvearrowright$ $\mathrm{Sub}(G)$ ⇐ compact metrizable totally disconnected space

$H \xrightarrow{g} H^g = g^{-1} H g$ adjoint action

$rk_{CB}(\mathrm{Sub}(G))$ – Cantor-Bendixon rank

$\mathcal{K}(G) \subset \mathrm{Sub}(G)$ – perfect kernel ($= \emptyset$ or Cantor set)

IRS (invariant random subgroup) is G-invariant probability measure on $\mathrm{Sub}(G)$.

General problem: For interesting groups describe simplex of IRG. Mostly interested in ergodic continuous measures.

(they are supported on $\mathcal{K}(G)$). For what groups such measures exist.

Th. [Bartholdi, Gr]. Every weakly branch groups has ergodic continuous IRS.

$$G \curvearrowright (X,\mu) \longrightarrow G \curvearrowright (\beta(X), \beta_*\mu)$$

$$\beta : x \longrightarrow St_G(x), \quad x \in X$$

if μ is $\underbrace{\text{continuous}}_{\text{ergodic}}$ and action is totally nonfree then $\beta_*\mu$ is ergodic continuous IRS on G

iRS on $\mathcal{L}_{n,p}$

$$\mathcal{K}(\mathcal{L}_{n,p}) = \mathrm{Sub}(\mathcal{A}_n), \quad \mathcal{A}_n = \bigoplus_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})^n$$

Th. [L. Bowen, Gr., Kravchenko] Let $M_*(\mathcal{L}_{n,p})$ be the simplex of iRS supported on $\mathrm{Sub}(\mathcal{A}_n)$. Then $M_*(\mathcal{L}_n)$ is a Poulsen symplex.

A simplex is called a Poulsen symplex if the set of its extreme points is dense. [it is unique up to affine isomorphism].

$\Longrightarrow$ have a "zoo" of iRS on Lamplighter type groups