Searching for a Secure Public Key Cryptosystem

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Outline of this talk

1. Attacks on Textbook RSA
2. Formalising security
3. Some lessons
4. A critique of a group-based PKC

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Generate $k/2$-bit primes p, q. Set $n = pq$.

Choose a k-bit integer e coprime to $(p-1)(q-1)$.

Set $d = e^{-1} \bmod (p-1)(q-1)$.

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If you can factor n, you can derive d from n and e. The scheme is then broken.

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You need a carefully designed randomised message padding scheme.

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Attack 3: If m is short, $c = m^e$ (no modular reduction). Decrypt by taking real eth roots.

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For all pairs (pk, sk) output by `KeyGen`, and all messages m,

$$\mathtt{Dec}_{pk,sk}\,\mathtt{Enc}_{pk}(m) = m.$$

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Depending on the security model, `Adv` might have other inputs (e.g. a challenge ciphertext).

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`Adv` wins the game if $b = b'$. The PKC is IND-CCA2 secure if for all poly time adversaries and all polynimials p

$$\text{Prob}(\texttt{Adv wins}) < 1/2 + 1/p(k)$$

for sufficiently large k.

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  - Submitting random ciphertexts gives no useful information.

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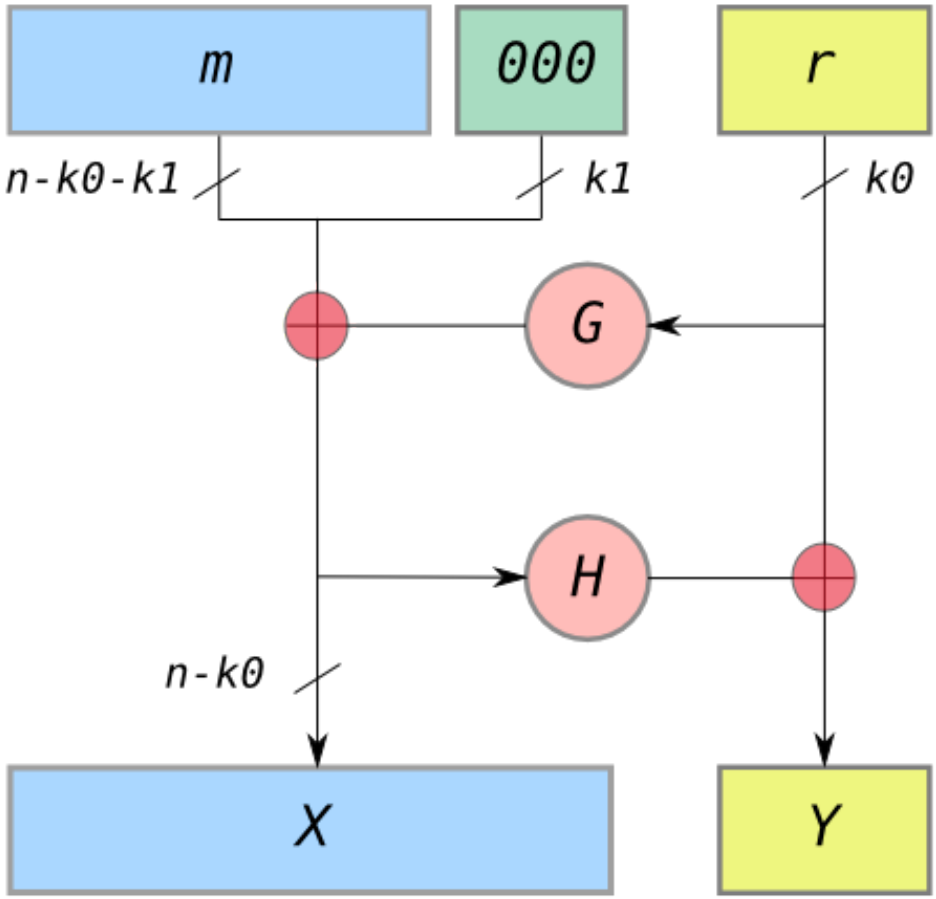
- Checks padded messages are correctly constructed.
- RSA-OAEP is IND-CCA2 secure in the random oracle model, provided taking eth roots modulo n is hard.
- Similar padding schemes exist for discrete log schemes.
- Generic methods (Fujisaki–Okamoto) exist too.

OAEP

... is defined as follows (G, H are hash functions):



Attribution for diagram: Ozga at en.wikipedia

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- The representation counts: think like a computer.
- Security is subtle. Cryptographers have thought hard about security for 35 years.
- So it's hard to say something new and interesting about security.
- Inventing new primitives (such as one-way functions) that cryptographers know how to build into cryptographic systems will go down well.

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Good example

Cryptographers love braid groups!

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Let's ignore the issue of making this precise.

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Shannon's Maxim is violated here.

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- But how can this be made rigorous? (Is computing an equivalent problem OK?)
- We could say that the adversary is not given the public key: but isn't this a symmetric key security model? (Then why not use the one-time pad?)
- Maybe the adversary is just given an encryption oracle. A highly non-standard assumption for a public key system!

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This can be solved by simulating Bob's encryption algorithm.

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  Non-standard security models are a (fairly) common problem with group-based cryptosystems.
- *Motivation is problematic.* Where is the link to an undecidable problem, as promised?
- For the above reasons, mainstream cryptographers are not likely to think this cryptosystem is important.

Some Links

This talk will appear soon on my home page:

`http://www.ma.rhul.ac.uk/sblackburn`

S.R. Blackburn, C. Cid and C. Mullan, 'Group theory in cryptography', Groups St Andrews 2009 in Bath, Volume 1 (CUP, 2011) 133-149.

`http://arxiv.org/abs/0906.5545`

N. Smart, 'Cryptography, an Introduction', 3rd edition.

`http://www.cs.bris.ac.uk/~nigel/`