Fully Homomorphic Encryption from LWE

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(Stanford)

Based on joint works with:

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(University of Toronto)

Craig Gentry

(IBM)

Post-Quantum Webinar,
November 2011

Outsourcing Computation

Outsourcing Computation

x

$f(x)$

Function

f

Outsourcing Computation

It's everywhere!

x

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search query

$f(x)$

Search results

Function

f

Google search

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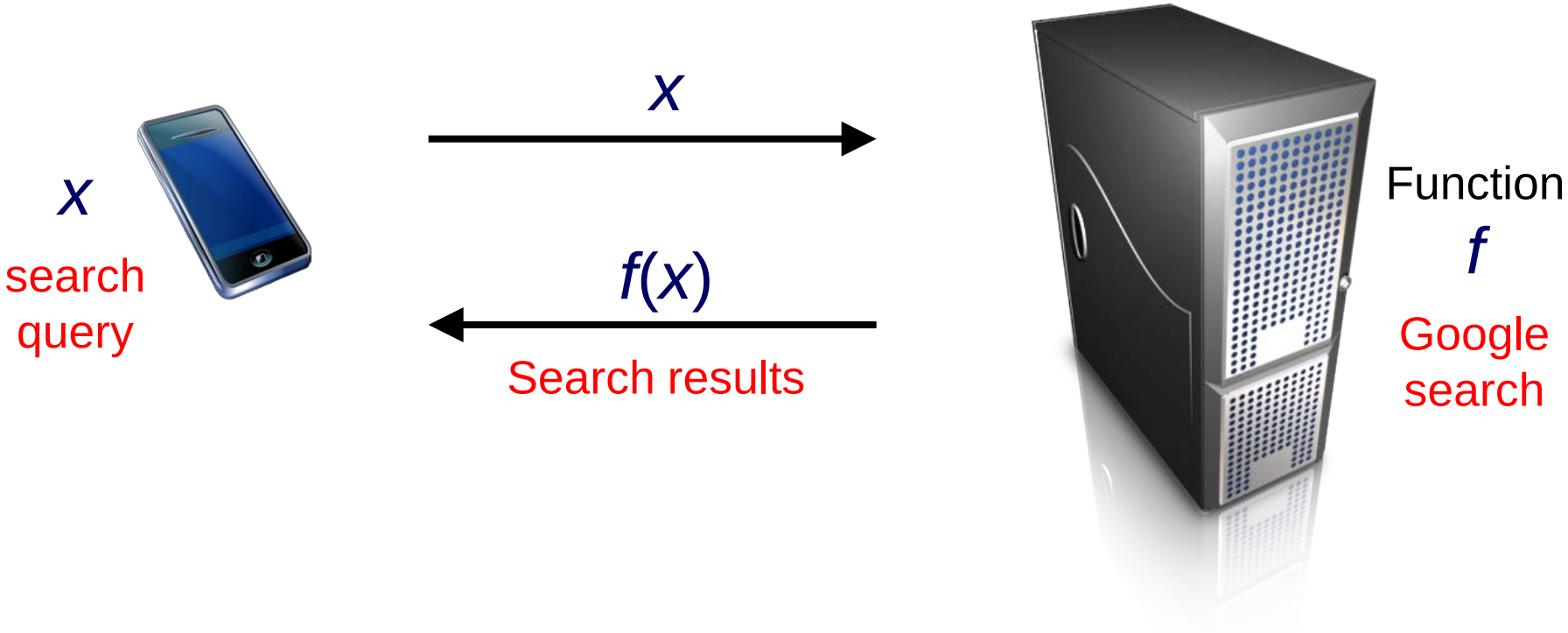

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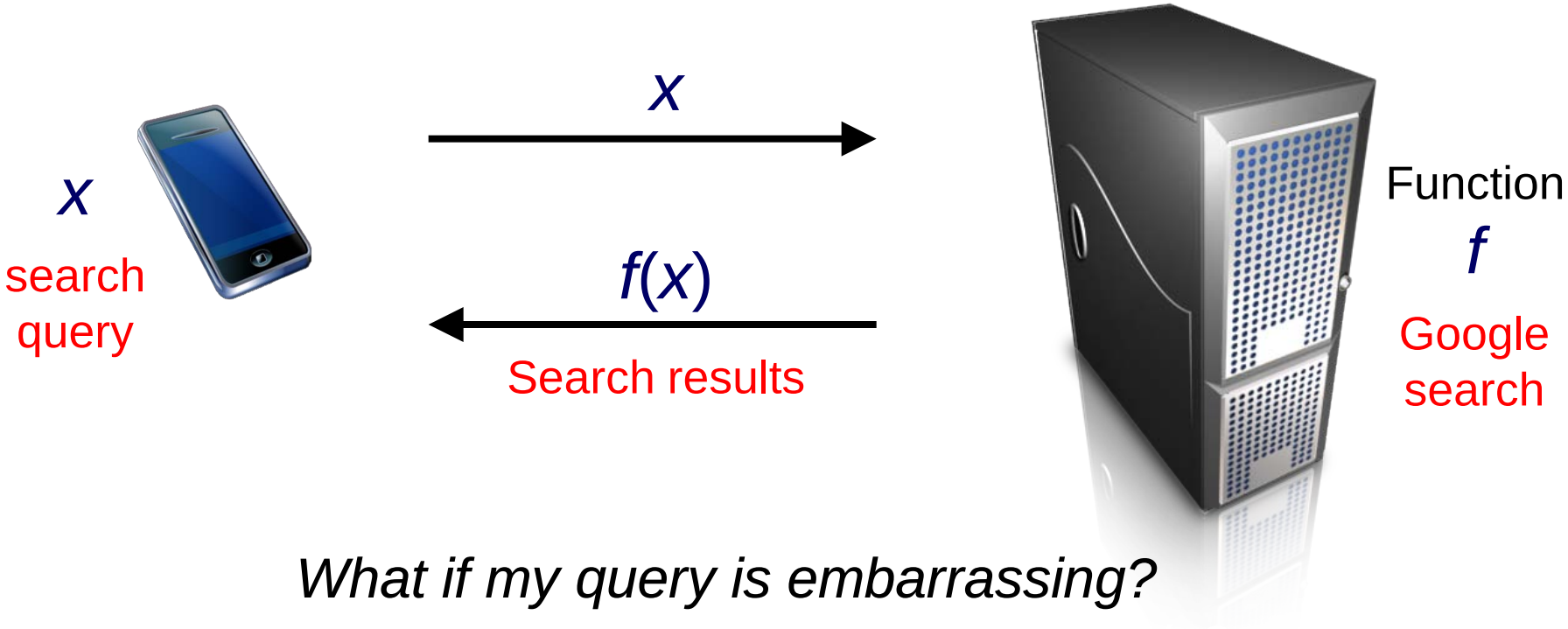

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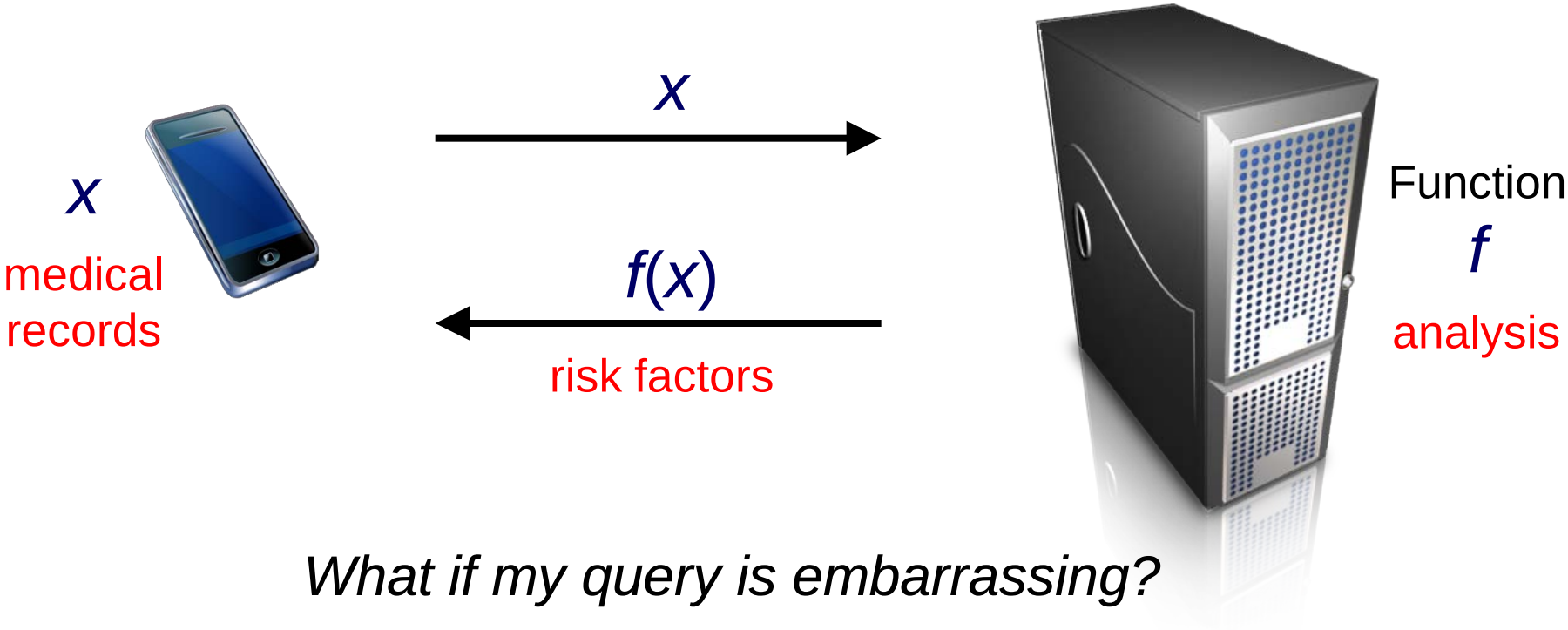

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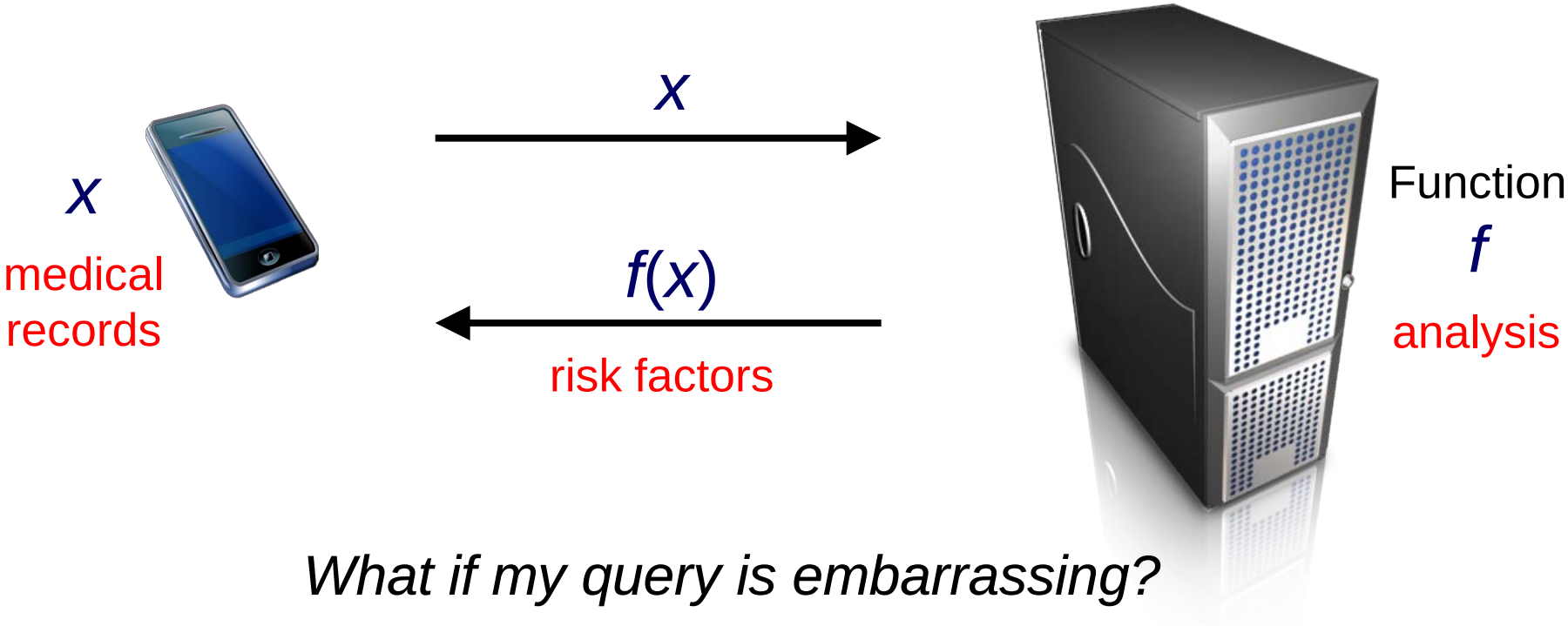

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Want Privacy!

Outsourcing Computation – Privately

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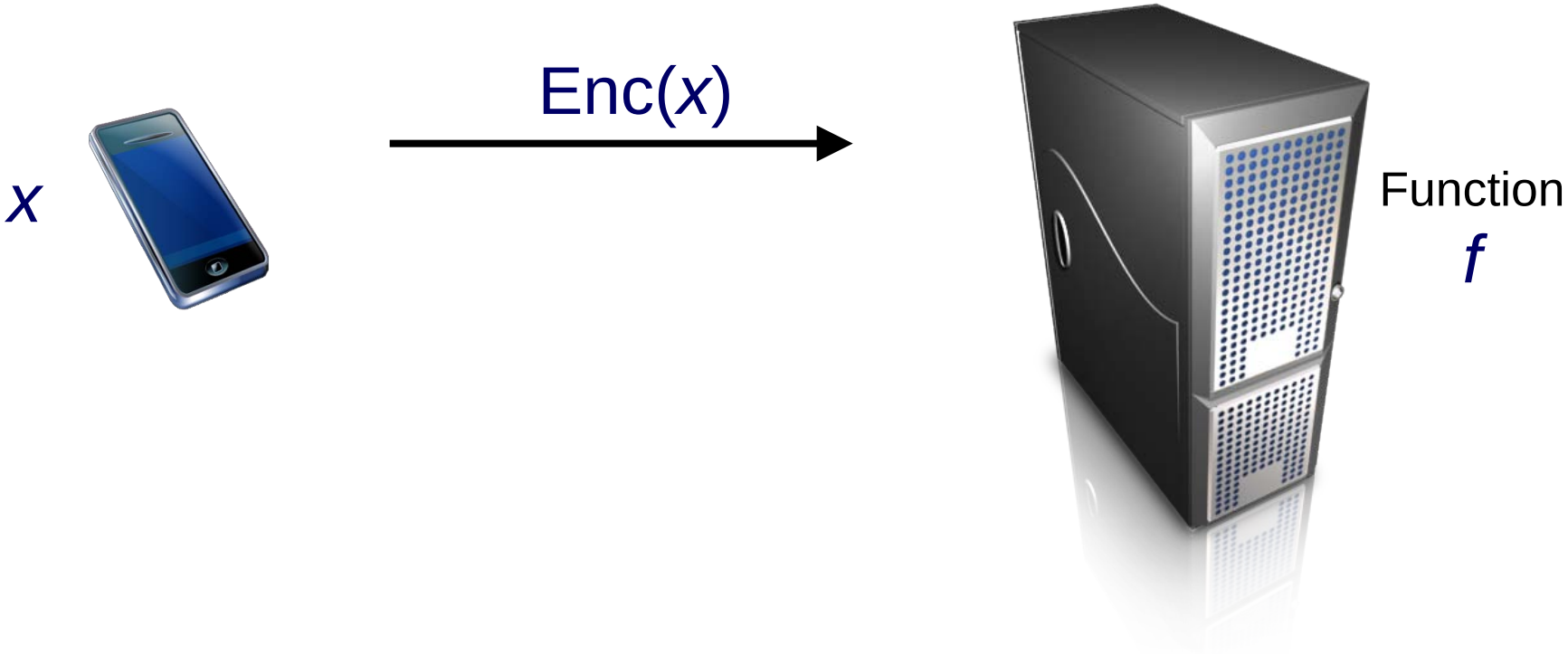

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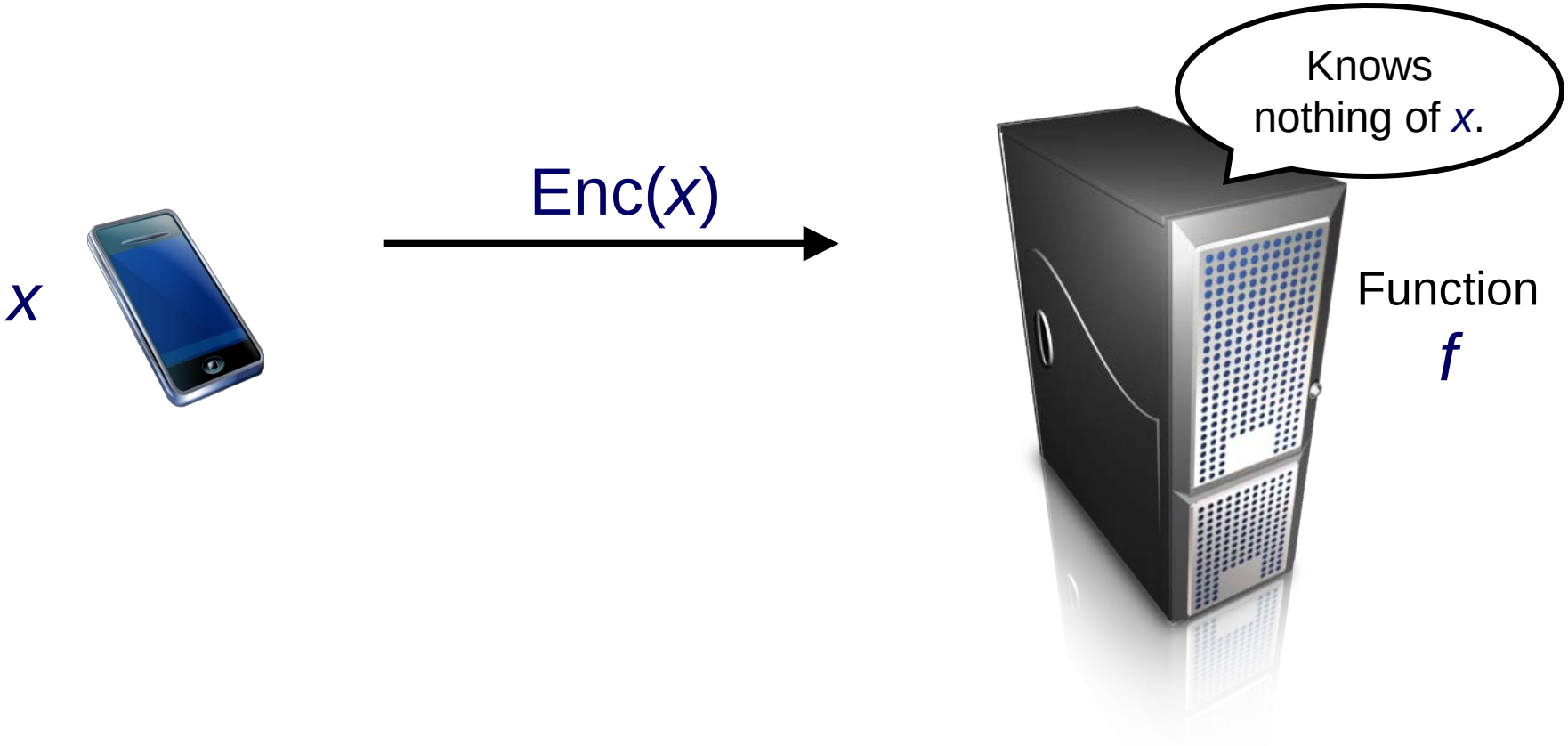

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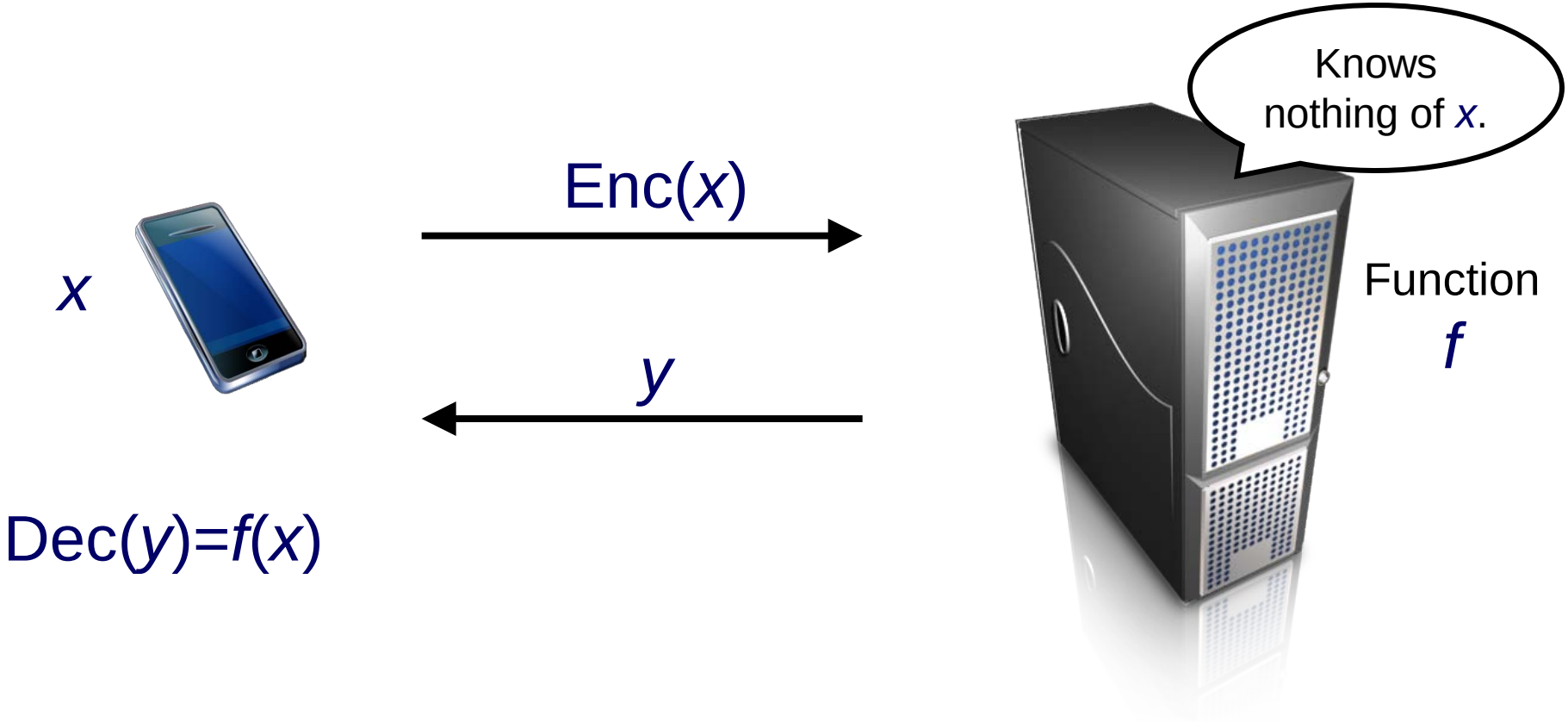

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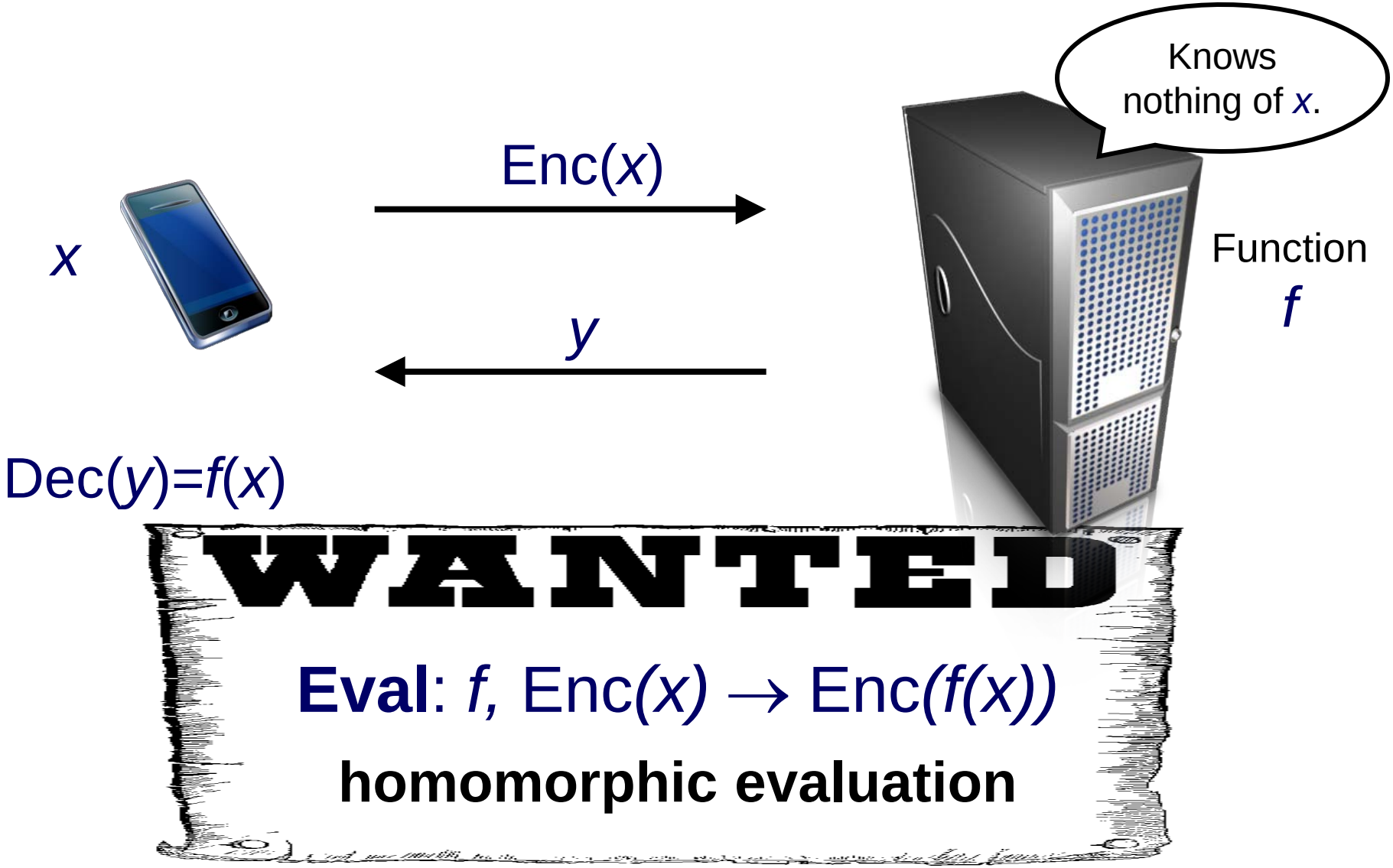

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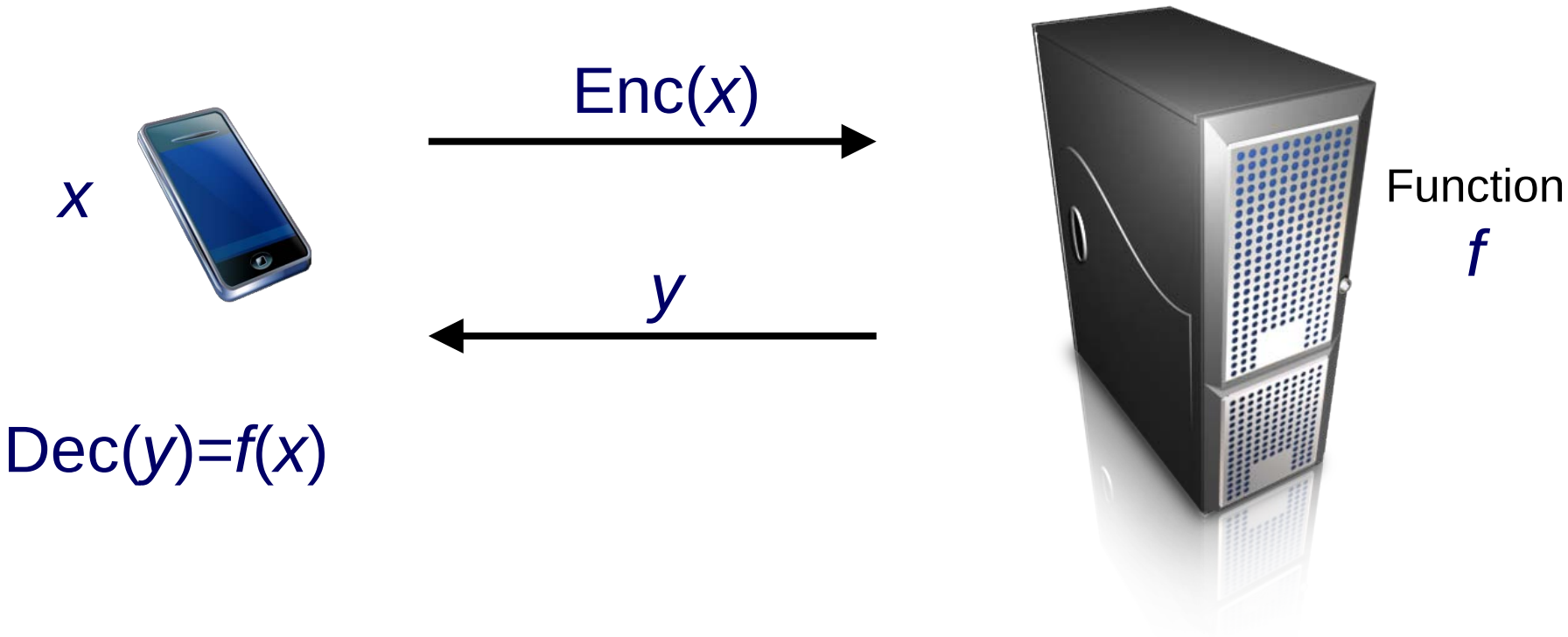

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evaluation key

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"Fully" = Evaluate all (efficient) f

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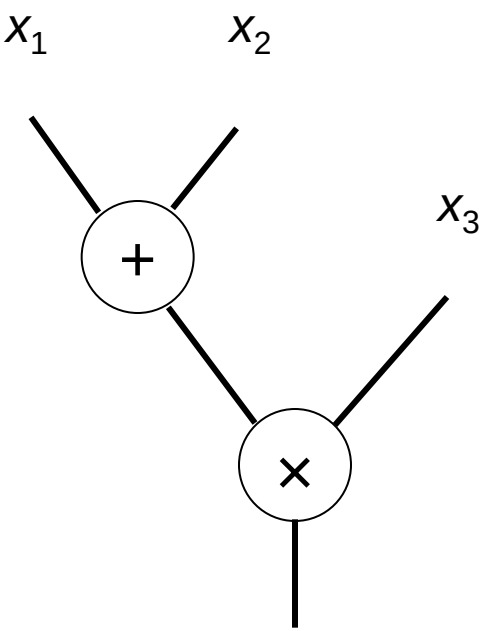
(a la [BGW88])

Arith. Circuit (+, ×) over GF(2).



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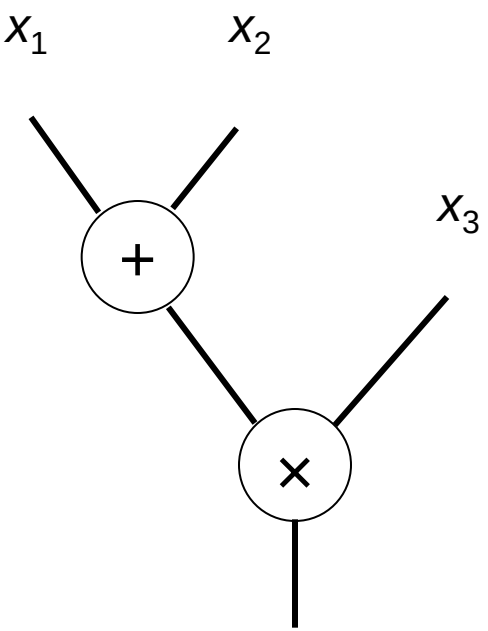

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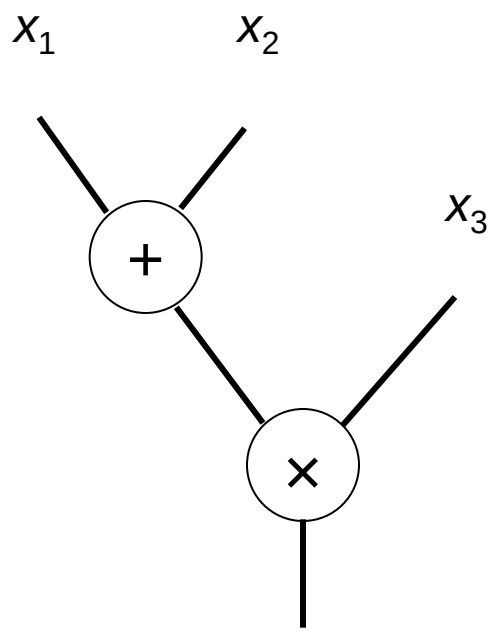

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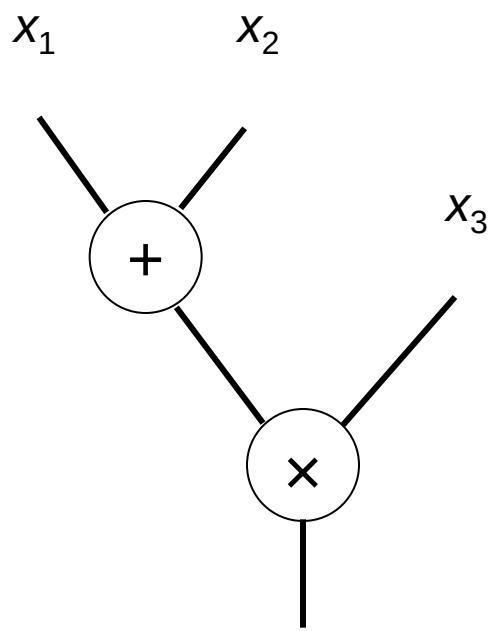

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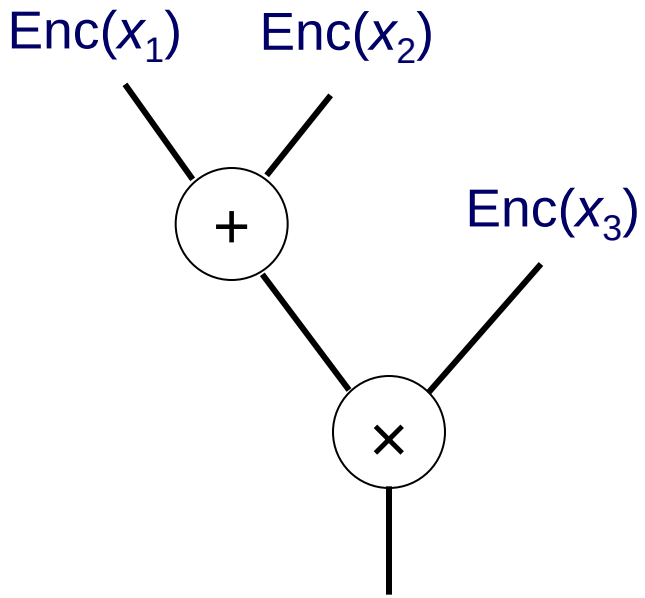

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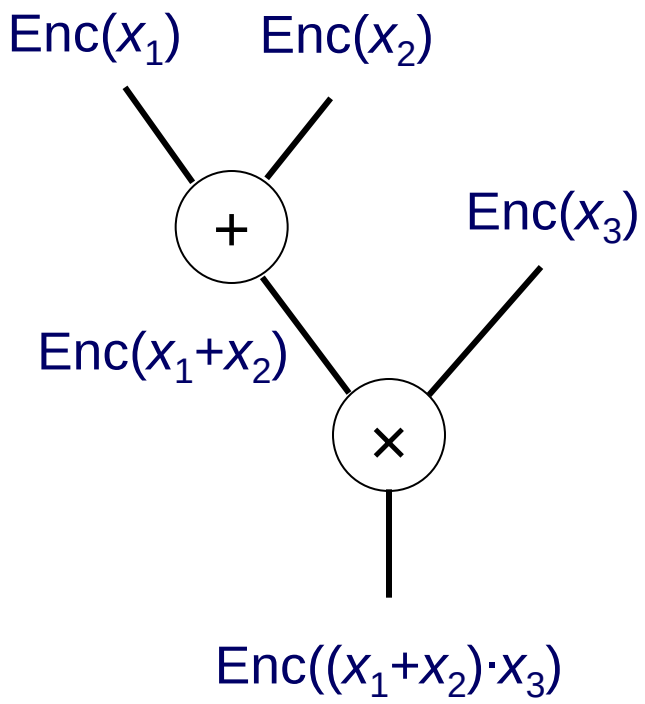

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Partial solutions:

- Only add [GM82,P99,R05,…].
- Only mult [G84,…].
- Add + single mult [BGN05,GHV10].
- Add + mult (w/ ciphertext blowup) [SYY99,GHV10,MGH10].

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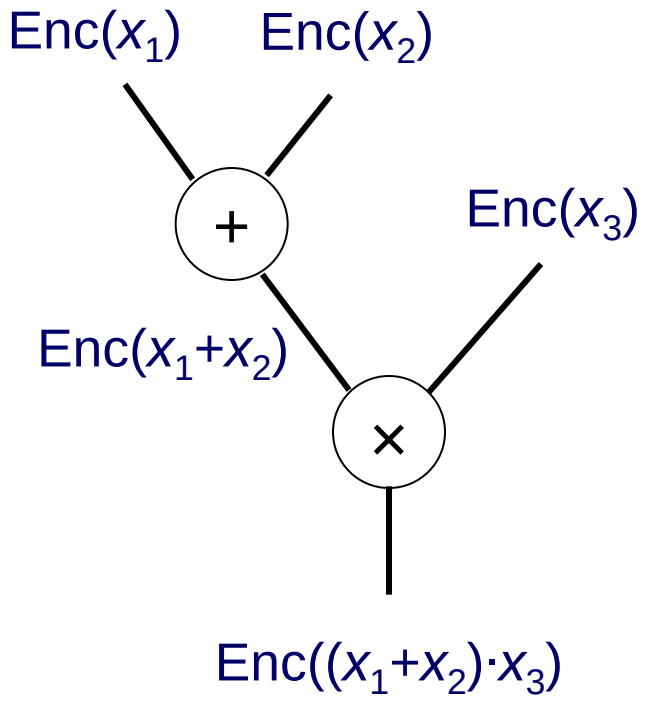

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"Deep enough" HE $\Rightarrow$ **FHE**

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Deep enough = Deeper than decryption circuit

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Eval for any depth *d* circuit (aka "somewhat" HE)

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Circular security assumption

Since Gentry

- Additional candidate FHE schemes:
  - [vDGHV10]: Approx. GCD + sparse subset sum (via squashing).
  - [BV11a]:
    - Ring-LWE + sparse subset sum (via squashing) / "sparse-ring-LWE".
    - Circular secure *d*-HE (not bootstrappable).
- Efficiency improvements of Gentry's scheme [SV10, SS10, GH11].

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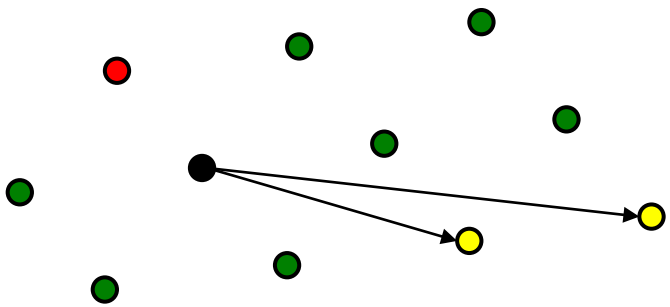
Short-vectors in lattice:



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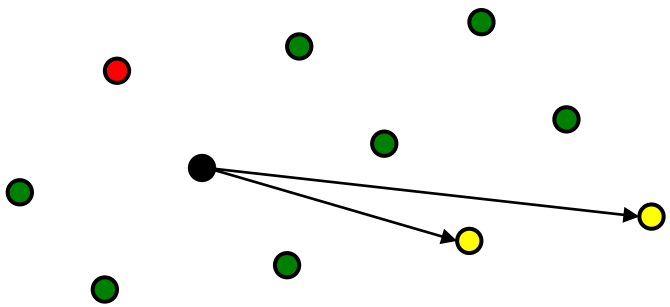

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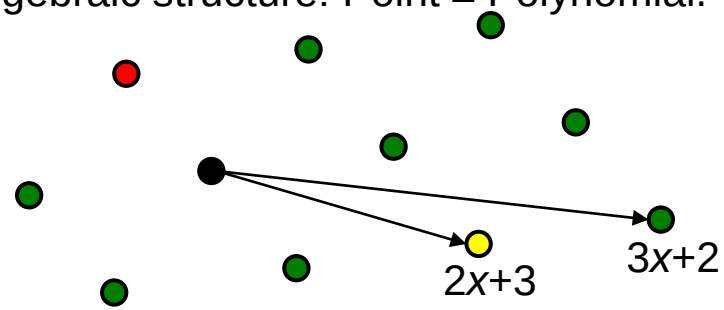

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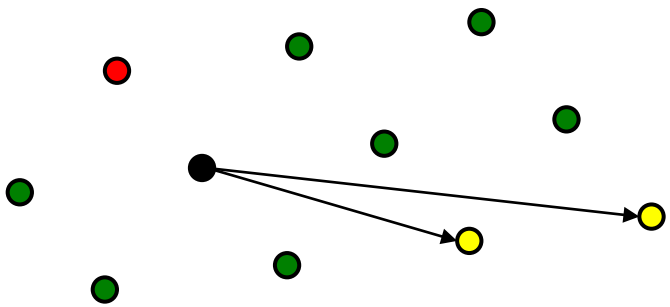

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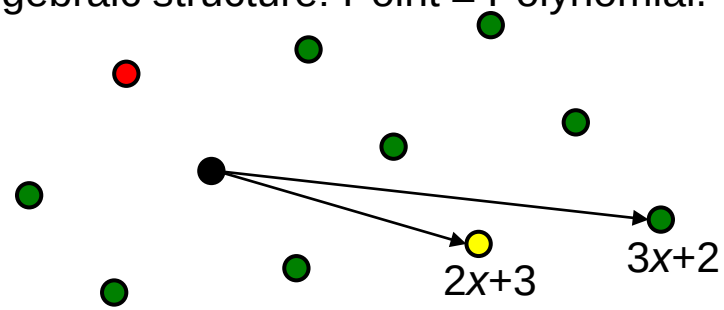

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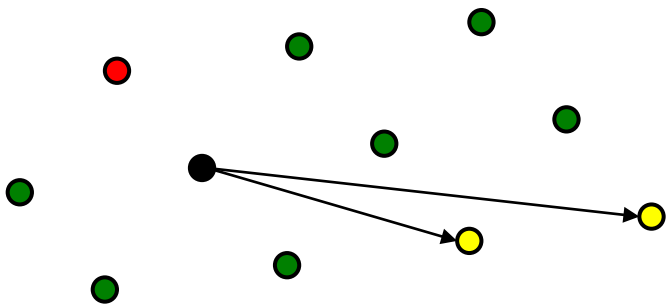

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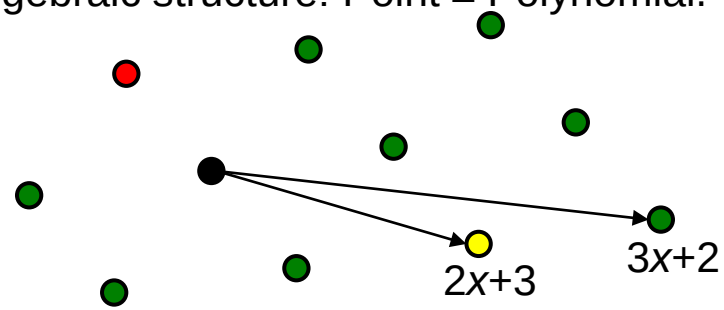

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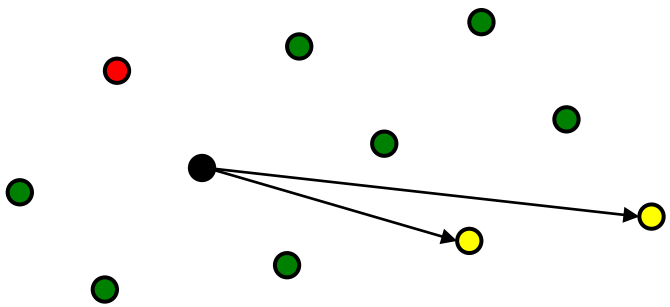

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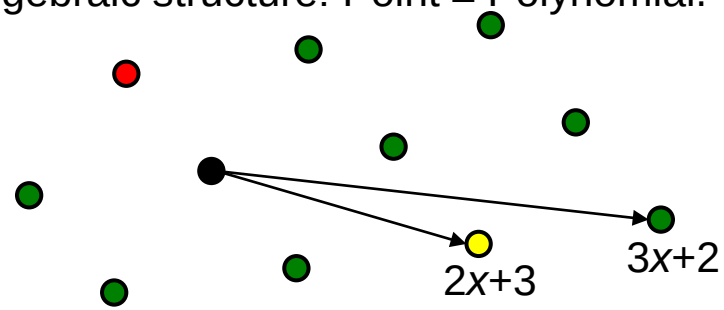

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(Are ideal assumptions inherent to FHE?)

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Concurrently [GH11]: Remove squashing under ideal lattice assumption (using arith. representation of dec. circuit).

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Short eval. key without additional assumption?

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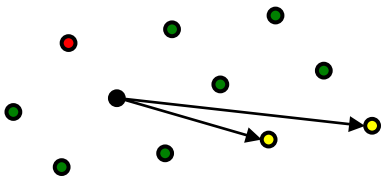
Fully Homomorphic Encryption



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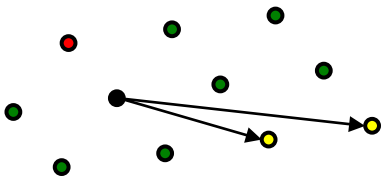

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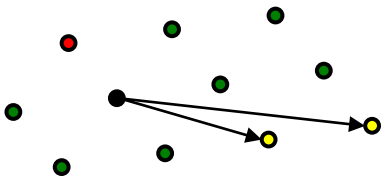

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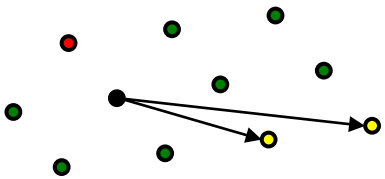

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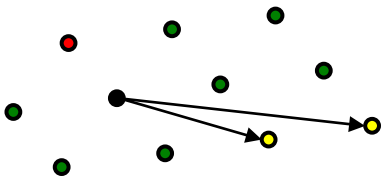

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Leveled FHE without bootstrapping.

- Conceptual contribution, although less efficient and worse parameters.

Our Results: Private Information Retrieval (PIR)

- **PIR** = Oblivious retrieval from "huge" DB of size N. [CGKS95,KO97,CMS99]
  - Quality measure: Communication complexity.

- **Our protocol:** $\tilde{O}(\log(N))$-communication.
  - Trivial LB = $\log(N)$. We are nearly optimal!
  - Previous best schemes [CMS99,L05,GR05,G09] $\sim\log^3(N)$.

New Ideas

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- **Re-linearization.**

  $\Rightarrow$ “Shallow” multiplicative homomorphism.

- **(Dimension-)Modulus reduction.**

  $\Rightarrow$ Noise management for “deep” circuits.

  $\Rightarrow$ Short ciphertexts as by-product.

Talk Outline

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- The LWE assumption.
- Re-linearization and modulus reduction.
- FHE using bootstrapping.
- (Leveled) FHE without bootstrapping.
- PIR protocol.
- Conclusion and open problems.

Learning With Errors (LWE) [R05]

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LWE$_{n,q}$ **:** For random secret $\boldsymbol{s} \in \boldsymbol{Z}_q^n$, for any m=poly(n):
sample random $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m \in \boldsymbol{Z}_q^n$;
"small" noise $e_1,\ldots,e_m \in \mathcal{E}$ ($|e_i| \le T << q$).

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Learning With Errors (LWE) [R05]

"Hermite normal form"

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Worst-Case Hardness (quantum [R05] **/ classical** [P09]**):**

Solve LWE $\Rightarrow$ (q/T)-apx. for short-vector problems in *worst-case* lattices.

(Best known apx. factor: 2^n in poly time; or $2^{\sqrt{n}}$ in $2^{\sqrt{n}}$ time.)

Symmetric Encryption with LWE

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(omitting public-key part)

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- **KeyGen:**
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  - Sample random $\boldsymbol{a} \in \boldsymbol{Z}_q^n$ and noise e, output $(\boldsymbol{a}, b) \in \boldsymbol{Z}_q^n \times \boldsymbol{Z}_q$,

    where $b = \langle \boldsymbol{a}, \boldsymbol{s} \rangle + 2e + m \in \boldsymbol{Z}_q$.

  - **Semantic security by LWE:** $(\boldsymbol{a}, b) \cong (\boldsymbol{a}, u)$

    (odd modulus $q \Rightarrow 2$ is invertible in $\boldsymbol{Z}_q$)

- **Decryption:** $\mathrm{Dec}_{\boldsymbol{s}}(\boldsymbol{a}, b) = (\, b - \langle \boldsymbol{a}, \boldsymbol{s} \rangle \,) \pmod 2$.

  - **Correctness:** $b - \langle \boldsymbol{a}, \boldsymbol{s} \rangle = b - \sum a[\,i\,] \cdot s[\,i\,] = m + 2e$ (over $\boldsymbol{Z}_q$).

    $\Rightarrow$ decryption succeeds if $e < q/4$.

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Additive Homomorphism

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$(\boldsymbol{a'}, b')$

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Additive Homomorphism

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Add the ciphertexts: $(\boldsymbol{a}_{add}, b_{add}) = (\boldsymbol{a} + \boldsymbol{a'}, b+b')$

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$$h_0 + \sum h_i \, s[i] + \sum h_{i,j} \, s[i] s[j] = mm' + 2(2ee' + me' + m'e)$$

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The coefficients h_i, $h_{i,j}$ are computable from input ciphertexts $(\boldsymbol{a}, b)$, $(\boldsymbol{a'}, b')$.

What's the output ciphertext?

Multiplicative Homomorphism

What's the Ciphertext?

$$h_0 + \sum h_i\, s[i] + \sum h_{i,j}\, s[i]s[j] = mm' + 2e''$$

- **Ciphertext = Coefficients $\{h_i\}$, $\{h_{i,j}\}$?**
  - Decrypt with secret key $\boldsymbol{s}$:

$$h_0 + \sum h_i\, s[i] + \sum h_{i,j}\, s[i]s[j] \pmod 2$$

$$= mm' + 2e'' \pmod 2$$

$$= mm' \pmod 2.$$

- **Problem:** Ciphertext contains $\sim n^2$ elements.
  - Size blows up with number of multiplications.

Find a more compact representation?

Re-Linearization

$$h_0 + \sum h_i \, s[i] + \sum h_{i,j} \, s[i] s[j] = mm' + 2e''$$

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Find linear function of ***s*** that represents this quadratic func.

Re-Linearization

$$h_0 + \sum h_i \, s[i] + \sum h_{i,j} \, s[i] s[j] = mm' + 2e''$$

Find linear function ~~of **s**~~ of new secret ***s'*** that represents this quadratic func.

Re-Linearization

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New **KeyGen**:

- Sample $\boldsymbol{s}, \boldsymbol{s'} \in \boldsymbol{Z}_q^n$ and set $sk = (\boldsymbol{s}, \boldsymbol{s'})$.
- Evaluation key evk : sample $\boldsymbol{A}_{i,j}$, $E_{i,j}$

$$\forall i,j. \quad \left(\boldsymbol{A}_{i,j}\,,\, B_{i,j} = \langle \boldsymbol{A}_{i,j}\,,\, \boldsymbol{s'} \rangle + 2E_{i,j} + s[i]s[j]\right)$$

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- **We get:** $s[i]s[j] \approx B_{i,j} - \langle \boldsymbol{A}_{i,j}\,,\, \boldsymbol{s'} \rangle$

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Re-Linearization

$$h_0+\sum h_i\, s[i] + \sum h_{i,j}\, s[i]s[j] = mm'+2e''$$

Plug back into quadratic equation:

$$\underbrace{h_0+\sum h_i\,(B_i - \langle \boldsymbol{A}_i, \boldsymbol{s}'\rangle) + \sum h_{i,j}\,(B_{i,j} - \langle \boldsymbol{A}_{i,j}, \boldsymbol{s}'\rangle)}_{\text{Linear in } \boldsymbol{s}'} = mm'+2e'''$$

c func.

Nev

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LWE $\Rightarrow$ Security still holds.

- t:

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$$h_0 + \sum h_i \left(B_i - \langle \boldsymbol{A}_i , \boldsymbol{s}' \rangle\right) + \sum h_{i,j} \left(B_{i,j} - \langle \boldsymbol{A}_{i,j} , \boldsymbol{s}' \rangle\right) = mm' + 2e'''$$

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Regrouping, we can now define $(\boldsymbol{a}_{\text{mult}}, b_{\text{mult}})$:

$$\boldsymbol{a}_{\text{mult}} = \sum h_i \boldsymbol{A}_i + \sum h_{i,j} \boldsymbol{A}_{i,j} \in \boldsymbol{Z}_q^n$$

$$b_{\text{mult}} = h_0+\sum h_i B_i + \sum h_{i,j} B_{i,j} \in \boldsymbol{Z}_q$$

Correctness:

$$\begin{aligned}
\text{Dec}_{\boldsymbol{s'}}(\boldsymbol{a}_{\text{mult}}, b_{\text{mult}}) &= \left(b_{\text{mult}} - \langle \boldsymbol{a}_{\text{mult}}, \boldsymbol{s'}\rangle\right) \pmod 2 \\
&= \left(h_0+\sum h_i B_i + \sum h_{i,j} B_{i,j}\right) - \left\langle \left(\sum h_i \boldsymbol{A}_i + \sum h_{i,j} \boldsymbol{A}_{i,j}\right), \boldsymbol{s'}\right\rangle \pmod 2 \\
&= h_0+\sum h_i (B_i - \langle \boldsymbol{A}_i , \boldsymbol{s'}\rangle) + \sum h_{i,j} (B_{i,j} - \langle \boldsymbol{A}_{i,j} , \boldsymbol{s'}\rangle) \pmod 2 \\
&= m \cdot m' \pmod 2
\end{aligned}$$

Re-Linearization

$$h_0 + \sum h_i \left(B_i - \langle \boldsymbol{A}_i , \boldsymbol{s}' \rangle\right) + \sum h_{i,j} \left(B_{i,j} - \langle \boldsymbol{A}_{i,j} , \boldsymbol{s}' \rangle\right) = mm' + 2e'''$$

Single multiplication accomplished!

For greater depth, repeat using new secrets ***s''***, ***s'''***…

Re-Linearization

$$h_0+\sum h_i\,(B_i - \langle \boldsymbol{A}_i\,,\,\boldsymbol{s'}\rangle) + \sum h_{i,j}\,(B_{i,j} - \langle \boldsymbol{A}_{i,j}\,,\,\boldsymbol{s'}\rangle) = mm'+2e'''$$

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Problem: Noise amplitude grows from T to $\approx T^2$.

(Actually, from nT to $(nT)^2$.)

Modulus Reduction: Reducing Noise Amplitude

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Naïve approach: Divide everything by nT.

Modulus Reduction: Reducing Noise Amplitude

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Can you believe that this actually works?

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Noise amplitude after modulus reduction:

$$\underbrace{(nT)^2/(nT)}_{\text{scaled down "original" noise}} + \underbrace{nT}_{\text{rounding error}} = O(nT)$$

Re-linearization and Modulus Reduction – Recap

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- Eval for mult. depth = 1.
- Modulus reduced: $q \Rightarrow q/(nT)$.
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Deeper Circuit? Just repeat!

- Mult. depth = d.
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Ciphertext decryptable if:

$$nT < (q/(nT)^d)/4$$

$$\Rightarrow q > 4(nT)^{d+1} = n^{O(d)}$$

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Very much so! As hard as quasi-poly apx. to short vectors in arbitrary lattices! (Not known even in time $2^{n^{1-\epsilon}}$.)

FHE with Bootstrapping

- No “squashing” whatsoever.
- Ciphertext size: $n \log (nT)$.
  - Post-bootstrapping size “survives”.
- Decryption depth: $O(\log(n))$.
- Security: $\mathbf{LWE}_{n,q=n^{O(\log n)}}$
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No bootstrapping whatsoever!

Talk Outline

- The LWE assumption.
- Re-linearization and modulus reduction.
- FHE using bootstrapping.
- (Leveled) FHE without bootstrapping.
- PIR protocol.
- Conclusion and open problems.

Single-Server PIR [CGKS95,KO97,CMS99]

Index

$x \in [N]$

Database

DB

$|DB| = N \approx 2^n$

Single-Server PIR [CGKS95,KO97,CMS99]

sk

evk

eval. key

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$cc=|\text{”Enc”}(x)|+|y|$

Single-Server PIR [CGKS95,KO97,CMS99]

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eval. key

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FHE ⇒ PIR

Use our FHE naïvely:

$$cc = n \cdot \log(nT) \cdot \log(N) \approx \tilde{O}(\log^2 N)$$

Single-Server PIR [CGKS95,KO97,CMS99]

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- Enc(x) using different, more efficient, scheme.
- Hom. decrypt efficient ciphertext and use as before.
- Using known efficient schemes: $cc = \tilde{O}(\log N)$.

Single-Server PIR [CGKS95,KO97,CMS99]

sk, *sym*

Enc(*sym*), *evk*

Index

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$\mathrm{Enc}_{sym}(x)$ →

← $y = \text{“Eval”}(\boldsymbol{DB}, \text{“Enc”}(x))$

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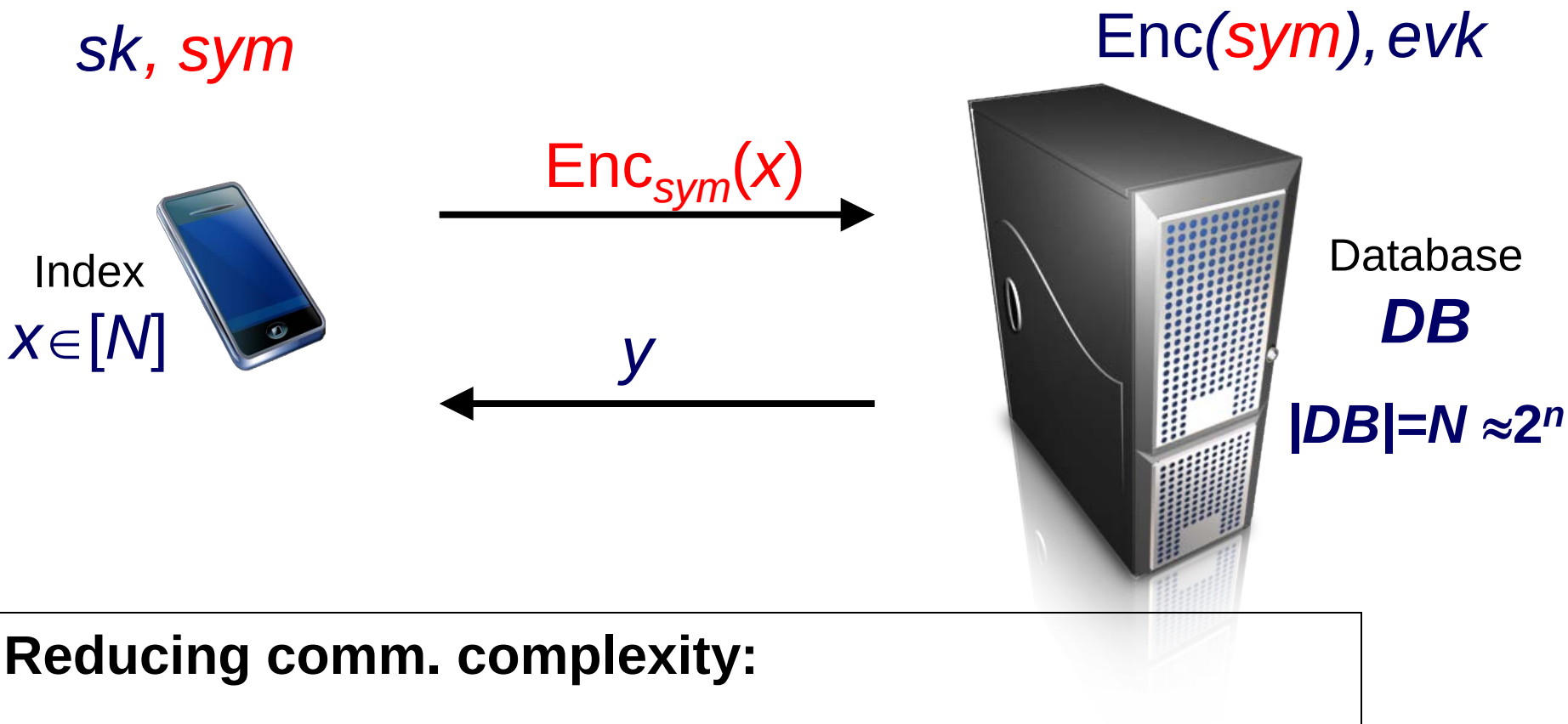

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Single-Server PIR [CGKS95,KO97,CMS99]

sk, sym

Enc(sym), evk

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$\mathrm{Enc}_{sym}(x)$ →

← y

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$\mathrm{Enc}_{sym}(x)+\mathrm{Enc}(sym) \Rightarrow \mathrm{Enc}(x)$

$y = \mathrm{Eval}(DB, \mathrm{Enc}(x))$

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Conclusion

- **FHE is not so complicated anymore…**
  - No ideals.
  - No squashing or sparse subset sum assumption.
  - **Efficiency!** Short ciphertexts, keys are easy to generate and are fairly unstructured.
  - **Can't lose:** Don't care about homomorpism? Use our scheme anyway – same efficiency and security.

- Private Information Retrieval.
  - Don't pay more than $\tilde{O}(\log(N))$.

Open Problems

- Remove circular security assumption.
- Go from quasi-polynomial to strictly polynomial LWE modulus.
- Improve efficiency, possibly using ideals [BGV11, LNV11, GHS11].
- New notions of security [BSW11].

Questions?